Geometric Satake for affine Lie algebras

HIRAKU NAKAJIMA

20210430

MIT Infinite Dimensional Algebra Seminar

<u>Plan</u> §1. Braverman-Finkelberg (-N) conjecture for geometric Satake for Kac-Moody via Coulomb branches

§2. Recall Arkhipov-Bezrukavnikov-Ginzburg, Ginzburg-Riche in usual geometric Satake

§3. Sketch of a proof for affine type A (joint work in progress with Dinakar Muthiah)

Apology In order to save my time, I will <u>not</u> properly refer contributions in usual geometric Satake (Lusztig, Ginzburg. Mirkovic-Vilonen, Beilinson-Drinfeld, ...)

§1.

$\mathfrak{g}^\vee$: symmetrizable Kac-Moody $\quad G^\vee$ adjoint type

$\mathfrak{g}$: Langlands dual

$Q = (Q_0, Q_1)$: valued quiver

V, W : Q_0-graded v.spaces $/\mathbb{C}$

$\lambda = \sum \dim W_i \cdot \Lambda_i$ dominant wt, $\mu = \lambda - \sum \dim V_i \alpha_i$ weight

Coulomb branch $\mathcal{M}_C(\lambda,\mu)$ symmetric BFN, valued N-Weekes

Assume $\mathfrak{g}$: symmetric

$G = \prod GL(V_i)$

$\mathbf{N} = \bigoplus \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus \mathrm{Hom}(W_i, V_i)$

Gr_G = affine Grassmannian of G

$\uparrow$

$\mathcal{R} = \{ [g(z), s(z)] \in G(\mathcal{K}) \times_{G(\mathcal{O})} \mathbf{N}(\mathcal{O}) \mid g(z)s(z) \in \mathbf{N}(\mathcal{O}) \}$

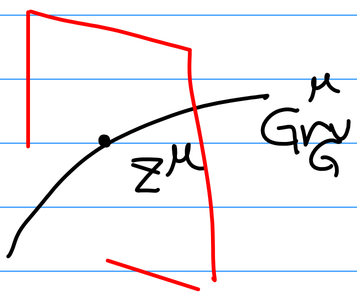
$H_*^{G(\mathcal{O})}(\mathcal{R})$ + convolution product commutative

$\mathcal{M}_C(\lambda,\mu) = \mathrm{Spec}$

<u>Th</u> [BFN, NW]

Suppose $\mathfrak{g}$: finite type, μ : <u>dominant</u>

$\mathcal{M}_C(\lambda,\mu)$ = slice to $\mathrm{Gr}^\mu_{G^\vee}$ in $\overline{\mathrm{Gr}^\lambda_{G^\vee}}$



$\mathcal{M}_C(\lambda, \mu)$ in general should be regarded as a slice for "Kac-Moody" affine Grassmannian

Rem. If $\mathfrak{g}^\vee$: untwisted affine $= \mathfrak{g}^\vee_{fin}$, μ : dominant

Conj. $\mathcal{M}_C(\lambda, \mu) =$ Uhlenbeck partial compactification of moduli space of $G^\vee_{fin}$-instantons on $\mathbb{C}^2/(\mathbb{Z}/\ell)$

$\ell = \text{level}(\lambda)$ $\quad$ $\mu =$ bdry condition
$\lambda =$ monodromy at 0.

True for type A [N-Takayama]

- original BF conj. was stated ↑ description.

$$T^\vee := \pi_1(\mathcal{G})^\wedge = \prod_i \pi_1(GL(V_i))^\wedge \curvearrowright \mathcal{M}_C(\lambda, \mu)$$

↑ max. torus of $G^\vee$

$$IC^\lambda_\mu = IC(\mathcal{M}_C(\lambda, \mu))$$

Conjecture
(0) $\mathcal{M}_C(\lambda, \mu)^{T^\vee}$ is either ϕ or a single pt z^μ

$\rho : \mathbb{C}^\times \to T^\vee$ regular, dominant coweight

$$\{z^\mu\} \xleftarrow{p} \underset{\text{set}}{\text{attracting}} \xrightarrow{j} \mathcal{M}_C(\lambda, \mu)$$

(1) $p_* j^! IC^\lambda_\mu$ is concentrated in degree 0

(2) $\bigoplus_\mu p_* j^! IC^\lambda_\mu$ has a structure of $\mathfrak{g}$-module
integrable h.w. $= \lambda$

(3) compatibility with restriction to std Levi subalgebra $\mathfrak{l}_J$ $\quad J \subset I = Q_0$

(4) tensor product is given by deformation or partial resolution of

$$\mathcal{M}_C(\lambda, \mu) \xleftarrow{\pi} \mathcal{M}^x_C(\lambda, \mu)$$

$\pi_* IC(\mathcal{M}^x_C(\lambda, \mu))$

<u>Th</u> [N]
True for affine type A.

$$a : \{z^\mu\} \hookrightarrow \mathcal{M}_C(\lambda, \mu)$$

<u>Conjecture</u> (5) $H^*(a^! IC^\lambda_\mu)$ vanishes in odd degree
and $\cong$ associated graded of $V(\lambda)_\mu$ with respect to Brylinski-Kostant filtration

$$F^i V(\lambda) = \{ v \in V(\lambda) \mid e^{i+1} v = 0 \}$$

e = principal nilpotent.

(6) $P^{\lambda}_{\mu}(q) = \sum q^i \dim gr^i_F V(\lambda)_\mu$

is given by Kostant partition function.

Slofstra 2010 (affine case)

(6) is not true, but true after

Correction F is replaced by $\mathring{F}$

$\mathring{F}^i V(\lambda) = \{ v \in V(\lambda) \mid x^{i+1} v = 0$ $\}$

$\forall x \in$ principal Heis $\cap\, n$

Th [Muthiah-N]

Conjecture (5) is true for affine type A.

§2.

Th [ABG 2004]

$\mathfrak{g}$: finite type, μ : dominant

$$H^0(T^*(G/B), \mathcal{O}(\mu)) \cong \bigoplus_{\lambda:\text{dominant}} V(\lambda)^* \otimes H^*(a^! IC^\lambda_\mu)$$

G-equiv. graded vector space isom.

$\mathbb{C}^\times$-action vs coh. degree

Moreover it is compatible with "product"

$\mathcal{O}(\mu_1) \otimes \mathcal{O}(\mu_2) \cong \mathcal{O}(\mu_1+\mu_2)$ vs convolution product.

This is a <u>precursor</u> of [BFN] Coulomb branch

$T^*G/B = G \times^B (\mathfrak{g}/\mathfrak{b})^*$

$H^0(T^*(G/B), \mathcal{O}(\mu)) = \mathrm{Ind}_B^G(\mathbb{C}[(\mathfrak{g}/\mathfrak{b})^*] \otimes \mathbb{C}_{-\mu})$

$\therefore\ H^*(a^! IC_\mu^\lambda) \cong \left(\mathrm{Res}_B^G V(\lambda) \otimes \quad '' \right)^B$

equivariant cohomology $\quad T^\vee \times \mathbb{C}^\times_{\mathrm{loop}} \curvearrowright \mathcal{M}_C(\lambda,\mu)$

given by cohomological degree

<u>Th</u> [GR 2013]

$\mathfrak{g}$: finite type

$$H^*_{T^\vee \times \mathbb{C}^\times_{\mathrm{loop}}}(a^! IC_\mu^\lambda) \cong \left(\mathrm{Res}_B^G V(\lambda) \otimes \mathbb{M}(\mu)\right)^B$$

$\mathbb{M}(\mu)$: asymptotic <u>universal</u> Verma module

$U_\hbar(\mathfrak{g})$: $xy - yx = \hbar[x,y]$

$(\underbrace{\mathbb{C}[\mathfrak{t}^\vee,\hbar]}_{\cong\, H^*_{T^\vee \times \mathbb{C}^\times_{\mathrm{loop}}}(pt)}, U_\hbar(\mathfrak{g}))$-bimodule

quantization of [ABG] on $T^*G/B + \mathcal{O}(\mu)$

<u>Remark</u> [GR] RHS explains BK filtration

$\longrightarrow$ proved (5).

Slofstra used RHS to show (6).
in the affine case

RHS makes sense for Kac-Moody setting

<u>Conj</u>. [GR] holds for any KM.

<u>Th</u> [MN] True for affine type A.

<u>Remark</u> In the affine case, RHS is a module of the coset VOA

If level $\lambda = 1$, then
$\mathfrak{g}$: affine ADE

① Arakawa–Creutzig–Linshaw
coset VOA = W–algebra

② $\bigoplus_\mu H^*_{T^\vee \times \mathbb{C}^\times}(\mathcal{A}, IC_\mu)$ is the Verma module
asym. universal
of W-algebra
(cf. AGT correspondence)

higher level λ <u>Conj</u> Belavin-Feigin
Nishioka – Tachikawa
as "higher level AGT"

§ Proof

Use the same strategy as [GR]
(used also in Coulomb branches)

$$
\begin{tikzcd}
H^*_{T^\vee \times \mathbb{C}^\times}(a^! IC^\lambda_\mu) \arrow[r, phantom, "\underset{?}{\cong}"] \arrow[d] & \left(\mathrm{Res}^G_B V(\lambda) \otimes \mathbb{M}(\mu)\right)^B \arrow[d] \\
H^*_{T^\vee \times \mathbb{C}^\times}(p_* j^! IC^\lambda_\mu) \arrow[r, phantom, "\cong"] & V(\lambda)_\mu \otimes \underbrace{\mathbb{C}[\mathfrak{t}^\vee, \hbar]}_{= H^*_{T^\vee \times \mathbb{C}^\times}(pt)}
\end{tikzcd}
$$

isomorphism
over $\mathrm{Frac}\, H^*_{T^\vee \times \mathbb{C}^\times}(pt)$
by localization

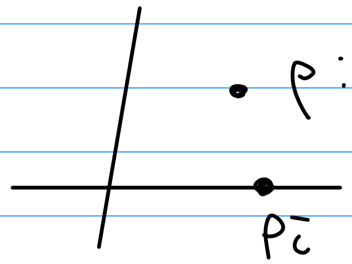
$\therefore \underset{?}{\cong}$ is defined
over Frac.

All four modules are free.

To check $\underset{?}{\cong}$ is defined over $H^*_{T^\vee \times \mathbb{C}^\times}(pt)$,
enough to check $\nexists$ pole in

possibly appears
in root hyperplanes

• simple root hyperplane



• $\rho : \mathbb{C}^\times \to T^\vee$

$(i \in Q_0)$

<u>Claim</u>. $\mathcal{M}_C(\lambda,\mu)^{\rho_i(\mathbb{C}^\times)} \cong$ Coulomb branch for $(\mathfrak{sl}_2)_i$

$\cong$ affine Grassmannian for $(PGL_2)_i$

Compatibility with restriction to std Levi.

$\longrightarrow$ OK

• real root hyperplane $\in$ Weyl(simpl)

Consider $w\rho$ $\quad w \in$ Weyl

Compare $p_* j^!$ and $p^w_* j^{w!}$ w.r.t. $w\rho$

<u>Claim</u> [BF another conj]

isom. given by localization is dynamical Weyl group

$\Longrightarrow$ no pole at real roots.

- imaginary root δ

$\rho_\delta : \mathbb{C}^\times \longrightarrow T^\vee$

contained in imaginary root hyperplane, but not in real hyperplanes

<u>Claim</u>:

$\mathcal{M}_C(\lambda,\mu)^{\rho_\delta(\mathbb{C}^\times)} \cong$ Coulomb branch for
$$\begin{tikzcd} n \arrow[loop right] \\ \boxed{\ell} \arrow[u] \end{tikzcd}$$
appropriate n

$$= S^n(\mathbb{C}^2/\mathbb{Z}/\ell)$$

$$\underline{\mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/\mathbb{Z}/\ell})} \longrightarrow$$

Representation theory side : restriction of $\mathfrak{g}$ to homogeneous <u>Heisenberg</u>.

$\longrightarrow$ OK for this case.

General KM

more general imaginary roots

$$\mathcal{M}_C\left(\begin{tikzcd} n \arrow[loop right] \arrow[loop above] \\ \boxed{\ell} \arrow[u] \end{tikzcd} \quad g\text{-loops}\right) = ?$$