Beilinson-Bernstein localization theorem, revisited, (2103.11193)

1) Localization theorem.

1.1) Statement.

G: s/simple algebraic group/$\mathbb{C}$

$T \subset B$: max. torus & Borel $\rightsquigarrow$

Lie algebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$

W: the Weyl group

$\mathcal{U} := \mathcal{U}(\mathfrak{g})$ universal enveloping algebra.

Harish-Chandra: center of $\mathcal{U} \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^W$

$\lambda \in \mathfrak{h}^* \rightsquigarrow$ max. ideal $\mathfrak{m}_\lambda \subset \mathbb{C}[\mathfrak{h}^*]^W \rightsquigarrow$

$\mathcal{U}_\lambda = \mathcal{U}/\mathcal{U}\mathfrak{m}_\lambda$ (so that $\mathcal{U}_\lambda = \mathcal{U}_{w\lambda}$)

– filtered algebra w. $\operatorname{gr} \mathcal{U}_\lambda = \mathbb{C}[\mathcal{N}]$,

where $\mathcal{N} \subset \mathfrak{g}^*$ is the nilpotent cone.

Goal of localization thm: relate rep'n theory of $\mathcal{U}_\lambda$ to the geometry of G/B.

$\lambda \in \mathfrak{h}^*$ $\rightsquigarrow$ sheaf of $(\lambda-\rho)$-twisted differ'l operators $\mathcal{D}^\lambda_{G/B}$ on G/B $(\rho = \frac{1}{2}\sum_{\alpha>0}\alpha)$

(for $\lambda=\rho$ get usual DO; for $\lambda \in$ wt. lattice $\rightsquigarrow$ DO in $\mathcal{O}_{G/B}(\lambda-\rho)$)

$\mathcal{D}^\lambda_{G/B}$ is a sheaf of filtered algebras on G/B w. $\operatorname{gr}\mathcal{D}^\lambda_{G/B} \xrightarrow{\sim} \eta_*\mathcal{O}_{T^*(G/B)}$

$(\eta: T^*(G/B) \twoheadrightarrow G/B)$

Fact (Beilinson-Bernstein):

$\Gamma(\mathcal{D}^\lambda_{G/B}) = \mathcal{U}_\lambda$ & $H^i(G/B, \mathcal{D}^\lambda_{G/B}) = 0$ $\forall i>0$.

$\rightsquigarrow$ adjoint global section & localization functors $\Gamma_\lambda: Coh(\mathcal{D}^\lambda_{G/B}) \rightleftarrows \mathcal{U}_\lambda\text{-mod}: \underbrace{Loc_\lambda}_{\mathcal{D}^\lambda_{G/B}\otimes_{\mathcal{U}_\lambda}\bullet}$

& $R\Gamma_\lambda: D^-(Coh(\mathcal{D}^\lambda_{G/B})) \leftrightarrows D^-(\mathcal{U}_\lambda\text{-mod}): LLoc_\lambda$

Theorem (Beilinson-Bernstein):

i) Γ_λ is exact (a.k.a. $R\Gamma_\lambda$ is t-exact) $\Leftrightarrow$ λ is *dominant* ($\langle\lambda,\alpha^\vee\rangle \notin \mathbb{Z}_{<0}$ $\forall$ root α).

ii) $R\Gamma_\lambda$ is equivalence $\Leftrightarrow$ λ is *regular* ($\langle\lambda,\alpha^\vee\rangle \neq 0$ $\forall$ root α).

iii) Γ_λ is equivalence $\Leftrightarrow$ λ is regular & dominant.

Goal of this talk: sketch a proof of most of this theorem that is way more complicated than original proof(s) but works in greater generality - of quantizations of conical symplectic resolutions.

1.2) Quantizations of conical symplectic resolutions.

Definition: A conical sympl'c resolution is a smooth variety X w. symplectic form ω & $\mathbb{C}^\times \curvearrowright X$ s.t.

- $\exists\, d \in \mathbb{Z}_{>0} \mid t.\omega = t^d\omega\ \forall\, t \in \mathbb{C}^\times$
- $\mathbb{C}[X]$ is fin. gen'd $\rightsquigarrow Y = \operatorname{Spec} \mathbb{C}[X]$ & $\rho: X \longrightarrow Y$
- ρ is projective & birational (hence resolution of singularities).
- $\mathbb{C}^\times$ contracts Y to a point.

Example: $X = T^*(G/B)$, $Y = \mathcal{N}$, ρ is Springer resolution, $\mathbb{C}^\times$ acts by fiberwise dilations.

More examples: (parabolic) Slodowy varieties, Nakajima quiver varieties, hypertoric var's.

$\mathbb{C}[Y]$ is a graded Poisson algebra so it makes sense to speak about its filtered quantizations.

Also can speak about filtered quantizations of $\mathcal{O}_X$. Those are classified by $H^2(X,\mathbb{C}) \ni \lambda \longleftrightarrow$ quantization $\mathcal{D}_\lambda$.

Example: $X = T^*(G/B) \Rightarrow H^2(X,\mathbb{C}) = \mathfrak{h}^*$
$\mathcal{D}_\lambda$ = (microlocalization) of $\mathcal{D}^\lambda_{G/B}$.

$\mathcal{A}_\lambda := \Gamma(\mathcal{D}_\lambda)$ is a quantization of $\mathbb{C}[Y]$ & $H^i(X,\mathcal{D}_\lambda) = 0 \rightsquigarrow$

$\Gamma_\lambda : Coh(\mathcal{D}_\lambda) \rightleftarrows \mathcal{A}_\lambda\text{-mod} : Loc_\lambda$ &

$R\Gamma_\lambda : D^-(Coh(\mathcal{D}_\lambda)) \rightleftarrows D^-(\mathcal{A}_\lambda\text{-mod}) : LLoc_\lambda$

Question: for which λ:

- is Γ_λ exact?
- is $R\Gamma_\lambda$ equivalence?
- is Γ_λ equivalence?

Approximate conjecture: • there's a finite collection of hyperplanes in $H^2(X,\mathbb{C})$ s.t. • $R\Gamma_\lambda$ is equivalence $\Leftrightarrow$ $\lambda \notin \bigcup$ these hyperplanes

• Γ_λ is equivalence $\Leftrightarrow$ $(\lambda + \text{ample classes}) \cap (\bigcup \text{these hyperplanes}) = \emptyset$

It's possible to describe hyperplanes in all examples. Techniques of this project allow to prove the conjecture in a number of examples incl. for Nakajima quiver varieties of finite & affine type A

2) Ideas of the proof:

2.1) Enlarge $\mathcal{U}_\lambda$ (in classic. case $X=T^*(G/B)$ – or $\mathcal{A}_\lambda$ in the general case) to an algebra $\tilde{\mathcal{U}}_\lambda$ w. idempotent $e\in\tilde{\mathcal{U}}_\lambda$ s.t.:

- $\mathcal{U}_\lambda \simeq e\tilde{\mathcal{U}}_\lambda e \rightsquigarrow e\bullet: \tilde{\mathcal{U}}_\lambda\text{-mod} \twoheadrightarrow \mathcal{U}_\lambda\text{-mod}$
- We have a left exact functor

$\tilde{\Gamma}_\lambda: Coh(\mathcal{D}^\lambda_{G/B}) \longrightarrow \tilde{\mathcal{U}}_\lambda\text{-mod}$ s.t.

(i) $\Gamma_\lambda \simeq e\tilde{\Gamma}_\lambda$

(ii) $R\tilde{\Gamma}_\lambda: D^b(Coh(\mathcal{D}^\lambda_{G/B})) \xrightarrow{\sim} D^b(\tilde{\mathcal{U}}_\lambda\text{-mod})\ \forall \lambda.$

We show that for λ regular & dominant $\tilde{\Gamma}_\lambda$ is equivalence $\overset{(i)}{\Rightarrow} \Gamma_\lambda$ is exact.

$\tilde{\mathcal{U}}_\lambda$ arises as quantization of endomorphism algebra of a tilting generator, $\mathcal{T}$, on X ($\Rightarrow$ (ii)). For (i) need $\mathcal{T}$ to have a line bundle summand. For $T^*(G/B)$ existence of $\mathcal{T}$ follows from [BMR].

2.2) Use categories $\mathcal{O}$.

At this point we know that:
λ is regular & dominant $\Rightarrow \Gamma_\lambda$ is exact
$(\Rightarrow \Gamma_\lambda \circ Loc_\lambda = id_{\mathcal{U}_\lambda\text{-mod}}) \Rightarrow \Gamma_\lambda$ is quotient functor $Coh(\mathcal{D}^\lambda_{G/B}) \twoheadrightarrow \mathcal{U}_\lambda\text{-mod}$.

Need to show Γ_λ is equivalence & first we do it on categories $\mathcal{O}$:
$\nu: \mathbb{C}^\times \to T$ generic $\rightsquigarrow \mathcal{O}_\nu(\mathcal{D}^\lambda_{G/B}) \subset Coh(\mathcal{D}^\lambda_{G/B})$
$\mathcal{O}_\nu(\mathcal{U}_\lambda) \subset \mathcal{U}_\lambda\text{-mod}$ w.

$$\Gamma_\lambda: \mathcal{O}_\nu(\mathcal{D}^\lambda_{G/B}) \twoheadrightarrow \mathcal{O}_\nu(\mathcal{U}_\lambda).$$

To show this restriction is equivalence, it's enough to show the categories $\mathcal{O}$ have the same number of simples:

$$\# \operatorname{Irr} \mathcal{O}_\nu(\mathcal{D}^\lambda_{G/B}) = |W| \quad \forall \lambda$$

$$\# \operatorname{Irr} \mathcal{O}_\nu(\mathcal{U}_\lambda) = |W| \ \forall \text{ reguler } \lambda$$

- easy!

For general sympl'c resolution X can talk about categories $\mathcal{O}$ if there's Hamiltonian torus $T \curvearrowright X$ w. $|X^T| < \infty$ (for $X = T^*(G/B)$ $\Rightarrow X^T \xrightarrow{\sim} W$).

Generic $\nu: \mathbb{C}^\times \longrightarrow T \rightsquigarrow$ categories $\mathcal{O}_\nu(\mathcal{D}_\lambda), \mathcal{O}_\nu(\mathcal{A}_\lambda)$

If Γ_λ is exact then $\Gamma_\lambda: \mathcal{O}_\nu(\mathcal{D}_\lambda) \twoheadrightarrow \mathcal{O}_\nu(\mathcal{A}_\lambda)$ and Γ_λ is equivalence between cat. $\mathcal{O}$

$\Longleftrightarrow \underbrace{\# \operatorname{Irr} \mathcal{O}_\nu(\mathcal{D}_\lambda)}_{|X^T|} = \# \operatorname{Irr} \mathcal{O}_\nu(\mathcal{A}_\lambda)$

We can analyze $\{\lambda \mid \Gamma_\lambda$ is exact & (essentially) $\# \operatorname{Irr} \mathcal{O}_\nu(\mathcal{A}_\lambda) = |X^T|\}$ but this is much harder and more technical than for $T^*(G/B)$.

2.3) From categories $\mathcal{O}$ to all modules.

For $T^*(G/B)$ to show $\Gamma_\lambda: \mathcal{O}_\nu(\mathcal{D}^\lambda_{G/B}) \xrightarrow{\sim} \mathcal{O}_\nu(\mathcal{U}_\lambda)$

$\Rightarrow \Gamma_\lambda: Coh(\mathcal{D}^\lambda_{G/B}) \xrightarrow{\sim} \mathcal{U}_\lambda\text{-mod}$

one uses Duflo's thm: every primitive ideal (:= annihilator of a simple module) in $\mathcal{U}_\lambda$ is the annihilator of a simple in $\mathcal{O}_\nu(\mathcal{U}_\lambda)$.

Duflo's thm is not available in gen'l ($T \curvearrowright X$ w. fin. many fixed pts) but there are sometimes other tools. E.g. for finite/affine type A Nakajima quiver var's all slices are again of that type & one can use some kind of induction.

Below we elaborate on 2.1) & 2.2)

3) Quantized endomorphism of tilting generators.

3.1) Tilting generators.

X is conical symplectic resolution (e.g. $T^*(G/B)$)

Definition: A tilting generator on X is a vector bundle $\mathcal{T}$ s.t.

(i) $Ext^i(\mathcal{T},\mathcal{T})=0 \ \forall i>0$

(ii) & $\tilde{A}:=End(\mathcal{T})$ has finite homological dimension

Corollary: $R\Gamma(\mathcal{T}\otimes\bullet): D^b(Coh(X)) \xrightarrow{\sim} D^b(\tilde{A}\text{-mod})$

Rem: Can assume $\mathcal{T}$ is $\mathbb{C}^\times$-equivariant $\Rightarrow \tilde{A}$ is graded.

(i) $\Rightarrow$ can uniquely deform $\mathcal{T}$ to a right $\mathcal{D}_\lambda$-module, $\mathcal{T}_\lambda \rightsquigarrow$

$\tilde{\mathcal{A}}_\lambda := End_{\mathcal{D}_\lambda}(\mathcal{T}_\lambda)$ - filtered deformation of $\tilde{A}$

$$\tilde{\Gamma}_\lambda = \Gamma(\mathcal{T}_\lambda\otimes\bullet): Coh(\mathcal{D}_\lambda) \to \tilde{\mathcal{A}}_\lambda\text{-mod}.$$

3.2) Property ♡.

We will need a certain condition on $\mathcal{T}$:

(♡) $\mathcal{T}$ has direct summand of rk 1

By twisting $\mathcal{T}$ w. line bundle can assume this summand is $\mathcal{O}_X \rightsquigarrow$ idempotent e s.t.

- $e\tilde{\mathcal{A}}_\lambda e = \mathcal{A}_\lambda$
- $\Gamma_\lambda \simeq e\tilde{\Gamma}_\lambda$

In particular, if $\tilde{\Gamma}_\lambda$ is equivalence $\Rightarrow \Gamma_\lambda$ is exact.

Here are examples when (♡) holds (for some choices of $\mathcal{T}$):

1) $X = T^*(G/B)$ (Bezrukavnikov-Mirkovic-Rumynin)

2) X is a symplectic resolution of symplectic quotient singularity V/Γ (Bezrukavnikov-Kaledin). For $\mathcal{T}$ can choose a "Procesi bundle") & $\tilde{A} = End(\mathcal{T}) = \mathbb{C}[V]\#\Gamma$, $\mathcal{A}_\lambda$ is symplectic reflection algebra.

3) X is a smooth version of the Coulomb branch of a gauge theory (Webster). These include finite/affine type A Nakajima quiver varieties.

Remark: These tilting generators are constructed starting from quantizations in characteristic p.

3.3) Localization for $\tilde{\mathcal{U}}_\lambda$.

- $R\Gamma(\mathcal{T}\otimes\cdot): D^b(\operatorname{Coh} X) \xrightarrow{\sim} D^b(\tilde{A}\text{-mod}) \Rightarrow$
$R\Gamma(\mathcal{T}_\lambda\otimes\cdot): D^b(\operatorname{Coh}\mathcal{D}_\lambda) \xrightarrow{\sim} D^b(\tilde{\mathcal{A}}_\lambda\text{-mod})$

- Γ_λ is equivalence $\Rightarrow$ $\mathcal{A}_\lambda$ & $\tilde{\mathcal{A}}_\lambda$ are Morita equivalent $\Rightarrow$ $\tilde{\Gamma}_\lambda$ is equivalence.

Theorem 1 (I.L.21) Assume (for simplicity $X = T^*(G/B)$)
Let λ be regular dominant. Then $\tilde{\Gamma}_\lambda$ is
an equivalence (can be generalized as long as
$(\heartsuit)$ holds - plus some technicalities).

Ideas of proof:

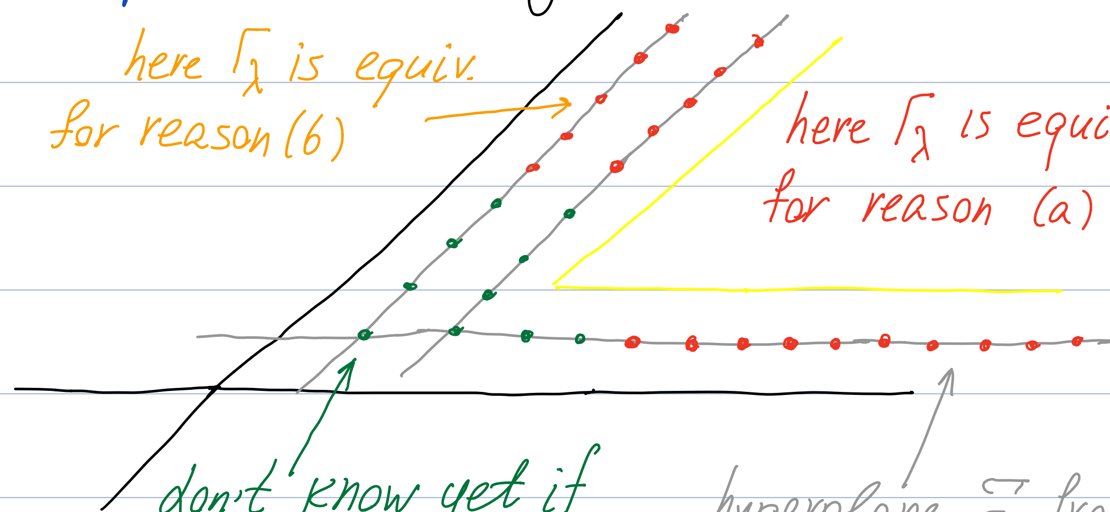
1) Identify a locus where Γ_λ is guaranteed
to be equivalence.

(a) deep enough inside the dominant
chamber (general reasons)

(b) let Ξ be a hyperplane intersecting
the dominant chamber and parallel to its wall,
deep enough in their intersection (from
reducing to the case of $T^*\mathbb{P}^1$)

(This works as stated for integral λ
but can be generalized by dropping "non-
essential" walls).

Example (rk 2, integral λ)



2) To handle the missing points (in green) use "translation bimodules"

$\tilde{\mathcal{U}}_{\lambda,\chi} \in \tilde{\mathcal{U}}_{\lambda+\chi}$-$\tilde{\mathcal{U}}_\lambda$-bimod, $\chi \in \Lambda$ (wt. lattice)

Construction (approximate, the actual is more technical)

$\chi \rightsquigarrow \mathcal{O}(\chi) \in \mathrm{Pic}(X) \rightsquigarrow$ quantization $\mathcal{D}_{\lambda,\chi}$ to $\mathcal{D}_{\lambda+\chi}$-$\mathcal{D}_\lambda$-bimodule $\rightsquigarrow$

$$\tilde{\mathcal{U}}_{\lambda,\chi} \underset{:=}{:\approx} \Gamma(\mathcal{T}_{\lambda+\chi} \otimes_{\mathcal{D}_{\lambda+\chi}} \mathcal{D}_{\lambda,\chi} \otimes_{\mathcal{D}_\lambda} \mathcal{T}_\lambda^*)$$

:= for dominant χ.

Role translation bimodules play:

- $\tilde{\mathcal{U}}_{\lambda,\chi} \otimes^L_{\tilde{\mathcal{U}}_\lambda} \bullet : D^b(\tilde{\mathcal{U}}_\lambda\text{-mod}) \xrightarrow{\sim} D^b(\tilde{\mathcal{U}}_{\lambda+\chi}\text{-mod})$
- Suppose $\tilde{\Gamma}_\lambda$ is an equivalence. TFAE
  - $\tilde{\Gamma}_{\lambda+\chi}$ is equivalence
  - $\tilde{\mathcal{U}}_{\lambda,\chi}$ is Morita equiv. bimodule
- for weights χ_1, χ_2 have homomorphism
$$\tilde{\mathcal{U}}_{\lambda+\chi_1,\chi_2} \otimes^L_{\tilde{\mathcal{U}}_{\lambda+\chi_1}} \tilde{\mathcal{U}}_{\lambda,\chi_1} \longrightarrow \tilde{\mathcal{U}}_{\lambda,\chi_1+\chi_2}$$

Under various conditions on λ, χ_1, χ_2 this is an isomorphism. Playing with these conditions & the previous two bullets leads to a proof of the theorem

$\square$

4) Categories $\mathcal{O}$.

4.1) Cartan subquotients.

X conical symplectic resolution, $T \curvearrowright X$ Hamiltonian torus action w. finitely many fixed points.

$\mathfrak{p} := H^2(X, \mathbb{C})$ - parameter space for quantizations $\rightsquigarrow$ universal quantizations $\mathcal{D}_{\mathfrak{p}}$ of X & $\mathcal{A}_{\mathfrak{p}}$ of $\mathbb{C}[X]$ - $\mathbb{C}[\mathfrak{p}]$-algebras with:

$\mathcal{A}_{\mathfrak{p}} = \Gamma(\mathcal{D}_{\mathfrak{p}})$, $\mathcal{D}_\lambda = \mathcal{D}_{\mathfrak{p}} \otimes_{\mathbb{C}[\mathfrak{p}]} \mathbb{C}_\lambda$, $\mathcal{A}_\lambda = \mathcal{A}_{\mathfrak{p}} \otimes_{\mathbb{C}[\mathfrak{p}]} \mathbb{C}_\lambda$.

Pick generic $\nu: \mathbb{C}^\times \longrightarrow T \rightsquigarrow$ gradings

$\mathcal{A}_{\mathfrak{p}} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_{\mathfrak{p}}^i$, $\mathcal{D}_{\mathfrak{p}} = \bigoplus_{i \in \mathbb{Z}} \mathcal{D}_{\mathfrak{p}}^i$, $\mathcal{A}_\lambda = \ldots$, $\mathcal{D}_\lambda = \ldots$

Definition: Cartan subquotients:

$C_\nu(\mathcal{A}_{\mathfrak{p}}) = \mathcal{A}_{\mathfrak{p}}^0 / \sum_{i>0} \mathcal{A}_{\mathfrak{p}}^{-i} \mathcal{A}_{\mathfrak{p}}^i$, $C_\nu(\mathcal{D}_{\mathfrak{p}})$, $C_\nu(\mathcal{A}_\lambda)$, $C_\nu(\mathcal{D}_\lambda)$.

Properties: $\cdot$ $C_\nu(\mathcal{D}_\mathfrak{p})$, $C_\nu(\mathcal{D}_\lambda)$ are sheaves on X^T (finite set). In fact,

$$C_\nu(\mathcal{D}_\mathfrak{p}) = \mathbb{C}[\mathfrak{p}]^{\oplus X^T}, \quad C_\nu(\mathcal{D}_\lambda) = \mathbb{C}^{\oplus X^T}.$$

$\cdot$ have natural homomorphism

$C_\nu(\mathcal{A}_\mathfrak{p}) \longrightarrow C_\nu(\mathcal{D}_\mathfrak{p})$ induced by $\mathcal{A}_\mathfrak{p} \xrightarrow{\sim} \Gamma(\mathcal{D}_\mathfrak{p})$.

$\cdot$ $C_\nu(\mathcal{A}_\lambda) = C_\nu(\mathcal{A}_\mathfrak{p}) \otimes_{\mathbb{C}[\mathfrak{p}]} \mathbb{C}_\lambda$

Example: $X = T^*(G/B)$, $\mathfrak{p} = \mathfrak{h}^*$, $\mathcal{D}_\mathfrak{p} = (\eta_* \mathcal{D}_{G/U})^T$
where $\eta : G/U \twoheadrightarrow G/B$

$\mathcal{A}_\mathfrak{p} = \mathbb{C}[\mathfrak{h}^*] \otimes_{\mathbb{C}[\mathfrak{h}^*]^W} \mathcal{U}$, $C_\nu(\mathcal{A}_\mathfrak{p}) = \mathbb{C}[\mathfrak{h}^*] \otimes_{\mathbb{C}[\mathfrak{h}^*]^W} \mathbb{C}[\mathfrak{h}^*]$.

$T^*(G/B)^T \xrightarrow{\sim} W \ni w \rightsquigarrow c_w : \mathfrak{h}^* \xhookrightarrow[x \mapsto (x, w(x))]{} \mathfrak{h}^* \times_{\mathfrak{h}^*/W} \mathfrak{h}^*$

The homomorphism

$C_\nu(\mathcal{A}_\mathfrak{p}) \longrightarrow C_\nu(\mathcal{D}_\mathfrak{p}) = \mathbb{C}[\mathfrak{h}^*]^{\oplus W}$ is $\bigoplus_{w \in W} c_w^*$.

4.2) Categories $\mathcal{O}$.

Definition: Category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ consists of fin. generated $\mathcal{A}_\lambda$-modules where $\mathcal{A}_\lambda^{>0} = \bigoplus_{i>0} \mathcal{A}_\lambda^i$ acts locally nilpotently.

Lemma: For a fin. gen'd $\mathcal{A}_\lambda$-module M TFAE

1) $M \in \mathcal{O}_\nu(\mathcal{A}_\lambda)$

2) M admits a (weakly) $\nu(\mathbb{C}^\times)$-equivariant str're & is supported on contracting locus for ν in Y.

Definition: Category $\mathcal{O}_\nu(\mathcal{D}_\lambda)$ is the full subcategory in $Coh(\mathcal{D}_\lambda)$ consisting of all objects that admit a $\nu(\mathbb{C}^\times)$-equivariant structure & are supported on contracting locus for ν in X.

Example: for ν dominant $\mathcal{O}_\nu(\mathcal{D}^\lambda_{G/B}) =$

 $= \{U\text{-equivariant } \mathcal{D}^\lambda_{G/B}\text{-modules}\}$.

Note that $\Gamma_\lambda, Loc_\lambda$ restrict to

$$\mathcal{O}_\nu(\mathcal{D}_\lambda) \rightleftarrows \mathcal{O}_\nu(\mathcal{A}_\lambda)$$

Lemma (description of simples)

- $\operatorname{Irr} \mathcal{O}_\nu(\mathcal{D}_\lambda) \xrightarrow{\sim} X^T \ (= \operatorname{Irr} \mathcal{C}_\nu(\mathcal{D}_\lambda))$
- $\operatorname{Irr} \mathcal{O}_\nu(\mathcal{A}_\lambda) \xrightarrow{\sim} \operatorname{Irr} \underbrace{\mathcal{C}_\nu(\mathcal{A}_\lambda)}_{\text{can be shown to have } \dim<\infty}.$

4.3) $\mathcal{O}$-regular parameters & localization.

Definition: Quantization parameter λ is *$\mathcal{O}$-regular* if $\mathcal{C}_\nu(\mathcal{A}_\lambda) \xrightarrow{\sim} \mathcal{C}_\nu(\mathcal{D}_\lambda)$.

– Zariski open locus in $H^2(X,\mathbb{C})$, $\neq \emptyset$.

Example: For $X = T^*(G/B)$, $\mathcal{O}$-regular $\Leftrightarrow$ regular.

Observation: If Γ_λ is exact $\Rightarrow \Gamma_\lambda \circ Loc_\lambda \simeq id$

$\Rightarrow \Gamma_\lambda : Coh(\mathcal{D}_\lambda) \twoheadrightarrow \mathcal{A}_\lambda\text{-mod}$ & $\mathcal{O}_\nu(\mathcal{D}_\lambda) \twoheadrightarrow \mathcal{O}_\nu(\mathcal{A}_\lambda)$ are Serre quotient functors.

So: if Γ_λ is exact & λ is $\mathcal{O}$-regular, then $\Gamma_\lambda: \mathcal{O}_\nu(\mathcal{D}_\lambda) \xrightarrow{\sim} \mathcal{O}_\nu(\mathcal{A}_\lambda)$ – Serre quotient functor between categories w. same (finite) number of simples.

Example: $X = T^*(G/B)$, λ is regular ($\Rightarrow \mathcal{O}$-regular) & dominant (so Γ_λ is exact by Theorem 1) Then $\Gamma_\lambda: \mathcal{O}_\nu(\mathcal{D}^\lambda_{G/B}) \xrightarrow{\sim} \mathcal{O}_\nu(\mathcal{A}_\lambda)$.

The case of general X is handled using:

Theorem 2 (I.L. 21) Let $\lambda \in H^2(X, \mathbb{C})$ be s.t. Γ_λ is exact. Then in a neighborhood of λ (in, say, usual topology) the complement to locus of $\mathcal{O}$-regular parameters has pure codim 1.