Beilinson-Bernstein localization theorem, revisited, (2103.11193)

1) Localization theorem.

1.1) Statement.

\(G: s/simple\ algebraic\ group/C\)
\(T \subset B: \text{max. torus} & \text{Borel} \sim\)

Lie algebras \(\mathfrak{g} \subset \mathfrak{b} \subset \mathfrak{c}\)

\(W: \text{the Weyl group}\)

\(U: = U(\mathfrak{g}) \text{ universal enveloping algebra}\).

Harish-Chandra: center of \(U \sim \mathbb{C}[\mathfrak{g}^*]^W\)

\(\lambda \in \mathfrak{g}^* \sim \text{max. ideal } \mathfrak{m}_\lambda \subset \mathbb{C}[\mathfrak{g}^*]^W \sim\)

\(U_\lambda = U/U_\lambda \quad (\text{so that } U_\lambda \sim U_{w_\lambda})\)

- filtered algebra w. gr \(U_\lambda \sim \mathbb{C}[N],\)
where \(N \subset \mathfrak{g}^*\) is the nilpotent cone.
Goal of localization thm: relate rep’n theory of $U_\lambda$ to the geometry of $G/B$.

$\lambda \in \mathfrak{h}^*$  $\rightsquigarrow$ sheaf of $(\lambda - \rho)$-twisted differential operators $\mathcal{D}^\lambda_{G/B}$ on $G/B$ ($\rho = \frac{1}{2} \sum_\alpha \alpha$).

(for $\lambda = \rho$ get usual $\mathcal{D}^\rho$; for $\lambda \in$ wt. lattice
$\rightsquigarrow \mathcal{D}^\rho$ in $\mathcal{O}_{G/B}(\lambda - \rho)$)

$\mathcal{D}^\lambda_{G/B}$ is a sheaf of filtered algebras
on $G/B$ w. $\text{gr } \mathcal{D}^\lambda_{G/B} \cong \mathfrak{g}^* \mathcal{O}_{T^*(G/B)}$

($\eta: T^*(G/B) \to G/B$)

Fact (Beilinson-Bernstein):

$\Gamma(\mathcal{D}^\lambda_{G/B}) = U_\lambda$ & $H^i(G/B, \mathcal{D}^\lambda_{G/B}) = 0 \quad \forall i > 0.$

$\rightsquigarrow$ adjoint global section & localization
functors $\Gamma_\lambda: \text{Coh}(\mathcal{D}^\lambda_{G/B}) \leftrightarrow U_\lambda\text{-mod: Loc}_\lambda$.

$\mathcal{D}^\lambda_{G/B} \otimes U_\lambda$.

$\& \text{R} \Gamma_\lambda: \text{D}(\text{Coh}(\mathcal{D}^\lambda_{G/B})) \leftrightarrow \text{D}(U_\lambda\text{-mod}): \text{LLoc}_\lambda$
Theorem (Beilinson-Bernstein):

1) $\Gamma_\lambda$ is exact (a.k.a. $R\Gamma_\lambda$ is t-exact) $\iff$

$\lambda$ is dominant ($<\lambda, d^\nu > \not\in \mathbb{Z}_0 + \text{root } \alpha$).

2) $R\Gamma_\lambda$ is equivalence $\iff$

$\lambda$ is regular ($<\lambda, d^\nu > \neq 0 + \text{root } \alpha$).

3) $\Gamma_\lambda$ is equivalence $\iff$

$\lambda$ is regular & dominant.

Goal of this talk: sketch a proof of most of this theorem that is way more complicated than original proof(s) but works in greater generality of quantizations of conical symplectic resolutions.
1.2) Quantizations of conical symplectic resolutions.

**Definition:** A conical symplectic resolution is a smooth variety $X$ w. symplectic form $\omega$ & $\mathbb{C}^* \times X$ s.t.

- $\exists d \in \mathbb{Z}_{>0} \mid t \cdot \omega = t^d \omega$ $\forall t \in \mathbb{C}^*$
- $\mathbb{C}[X]$ is fin. gen'd $\sim Y = \text{Spec } \mathbb{C}[X]$ & $p: X \to Y$
  - $p$ is projective & birational (hence resolution of singularities).
  - $\mathbb{C}^*$ contracts $Y$ to a point.

**Example:** $X = T^* (G/B)$, $Y = N$, $p$ is Springer resolution, $\mathbb{C}^*$ acts by fiberwise dilations.

More examples: (parabolic) Slodowy varieties, Nakajima quiver varieties, hypertoric var.’s.
$\mathcal{C}[y]$ is a graded Poisson algebra so it makes sense to speak about its filtered quantizations.
Also can speak about filtered quantizations of $Q_\chi$. Those are classified by $H^2(X, \mathbb{C}) \cong \lambda \leftrightarrow \text{quantization } D_\lambda$.

**Example:** $X = T^*(G/B) \Rightarrow H^2(X, \mathbb{C}) = \mathfrak{g}^\ast$

$D_\lambda = (\text{microlocalization})$ of $D_{G/B}$.

$\mathcal{A}_\lambda : = \Gamma(D_\lambda)$ is a quantization of $\mathcal{C}[y]$

& $H^i(X, D_\lambda) = 0 \rightsquigarrow$

\[ \Gamma_\lambda : \text{Coh}(D_\lambda) \leftrightarrow \mathcal{A}_\lambda\text{-mod} : \text{Loc}_\lambda \& \\
\text{R}\Gamma_\lambda : \mathcal{D}^{-}(\text{Coh}(D_\lambda)) \leftrightarrow \mathcal{D}^{-}(\mathcal{A}_\lambda\text{-mod}) : \mathcal{L}\text{Loc}_\lambda \]
Question: for which $\lambda$:

- is $\Gamma_\lambda$ exact?
- is $R\Gamma_\lambda$ equivalence?
- is $\Gamma_\lambda$ equivalence?

Approximate conjecture: there’s a finite collection of hyperplanes in $H^2(X, \mathbb{C})$

s.t. $R\Gamma_\lambda$ is equivalence $\iff$

$\lambda \notin U$ these hyperplanes

- $\Gamma_\lambda$ is equivalence $\iff$

$(\lambda + \text{ample classes}) \cap (U \text{ these hyperplanes}) = \emptyset$

It’s possible to describe hyperplanes in all examples. Techniques of this project allow to prove the conjecture in a number of examples incl. for Nakajima quiver varieties of finite & affine type A
2) **Ideas of the proof:**

2.1) **Enlarge** $U_\lambda$ (in classic case $X = T^*(G/B)$ — or $A_\lambda$ in the general case) to an algebra $\widetilde{U}_\lambda$ w. idempotent $e \in \widetilde{U}_\lambda$ s.t.:

- $U_\lambda \cong e \widetilde{U}_\lambda e \hookrightarrow e \cdot : \widetilde{U}_\lambda\text{-mod} \to U_\lambda\text{-mod}$
- We have a left exact functor $\overline{\Gamma}_\lambda : \text{Coh} (D^\lambda_{G/B}) \longrightarrow \widetilde{U}_\lambda\text{-mod}$ s.t.
  
  (i) $\overline{\Gamma}_\lambda \cong e \overline{\Gamma}_\lambda$
  
  (ii) $R\overline{\Gamma}_\lambda : D^b (\text{Coh} (D^\lambda_{G/B})) \to D^b (\widetilde{U}_\lambda\text{-mod}) \neq \lambda$.

We show that for $\lambda$ regular & dominant

$\overline{\Gamma}_\lambda$ is equivalence $\Rightarrow \overline{\Gamma}_\lambda$ is exact.

$\widetilde{U}_\lambda$ arises as quantization of endomorphism algebra of a tilting generator, $T$, on $X$ ($\Rightarrow (ii)$). For (i) need $T$ to have a line bundle summand. For $T^*(G/B)$ existence of $T$

follows from [BMR].
2.2) Use categories $O$.

At this point we know that:

- $\lambda$ is regular & dominant $\Rightarrow \Gamma^\lambda$ is exact
- $G^\lambda(\mathcal{O}_{G/B}) = \text{Coh}(\mathcal{D}^\lambda_{G/B})$ $\Rightarrow \Gamma^\lambda$ is quotient functor $\text{Coh}(\mathcal{D}^\lambda_{G/B}) \to \mathcal{U}^\lambda$-mod.

Need to show $\Gamma^\lambda$ is equivalence & first we do it on categories $O$:

$v: C^x \to T$ generic $\Rightarrow O^v(\mathcal{D}^\lambda_{G/B}) = \text{Coh}(\mathcal{D}^\lambda_{G/B})$

$O^v(\mathcal{U}^\lambda) = \mathcal{U}^\lambda$-mod w.

$\Gamma^\lambda: O^v(\mathcal{D}^\lambda_{G/B}) \to O^v(\mathcal{U}^\lambda)$

To show this restriction is equivalence, it's enough to show the categories $O$ have the same number of simples:

- $\# \text{Irr } O^v(\mathcal{D}^\lambda_{G/B}) = |W| \neq \lambda$
- $\# \text{Irr } O^v(\mathcal{U}^\lambda) = |W| \neq $ regular $\lambda$

- easy!
For general symplectic resolution $X$ one can talk about categories $\mathcal{O}$ if there's Hamiltonian torus $T \curvearrowright X$ w. $|X^T| < \infty$ (for $X = T^*(G/B)$

$\Rightarrow X^T \sim W$).

Generic $\lambda : C^* \rightarrow T \hookrightarrow$ categories $O^+_{\lambda}(D\lambda), O^+_{\lambda}(\mathfrak{h}_\lambda)$

If $\Gamma_\lambda$ is exact then $\Gamma_\lambda : O^+_{\lambda}(D\lambda) \rightarrow O^+_{\lambda}(\mathfrak{h}_\lambda)$ and $\Gamma_\lambda$ is equivalence between cat. $\mathcal{O}$

$\iff \# \text{Irr } O^+_{\lambda}(D\lambda) = \# \text{Irr } O^+_{\lambda}(\mathfrak{h}_\lambda)$

$|X^T|$

We can analyze $\{\lambda | \Gamma_\lambda \text{ is exact and }$

(essentially) $\# \text{Irr } O^+_{\lambda}(\mathfrak{h}_\lambda) = |X^T|^2$

but this is much harder and more technical than for $T^*(G/B)$.
2.3) From categories $O$ to all modules

For $T^*(G/B)$ to show $\Gamma_\lambda: O_\gamma(D_\lambda^{G/B}) \rightarrow O_\gamma(U_\lambda)$

$\Rightarrow \Gamma_\lambda: \text{Coh}(D_\lambda^{G/B}) \rightarrow U_\lambda$-$\text{mod}$

one uses Duflo's thm: every primitive ideal (:= annihilator of a simple module) in $U_\lambda$ is the annihilator of a simple in $O_\gamma(U_\lambda)$.

Duflo's thm is not available in general ($T\not\sim X$ w. fin. many fixed pts) but there are sometimes other tools. E.g. for finite/affine type A Nakajima quiver var's all slices are again of that type & one can use some kind of induction.

Below we elaborate on 2.1) & 2.2)
3) Quantized endomorphism of tilting generators

3.1) Tilting generators.

$X$ is conical symplectic resolution (e.g. $T^*(G/B)$)

**Definition:** A tilting generator on $X$ is a vector bundle $T$ s.t.

(i) $\text{Ext}^i(T, T) = 0 \quad \forall \ i > 0$

(ii) $\tilde{A} := \text{End}(T)$ has finite homological dimension

**Corollary:** $R\Gamma(T \otimes \cdot): D^b(\text{coh}(X)) \to D^b(\tilde{A} \text{-mod})$

**Rem:** Can assume $T$ is $\mathbb{C}^*$-equivariant

$\Rightarrow \tilde{A}$ is graded.

(i) $\Rightarrow$ can uniquely deform $T$ to a right $D^b_\lambda$-module: $T_\lambda \sim$

$\hat{\mathfrak{F}}_\lambda := \text{End}_{D^b_\lambda}(T_\lambda)$-filtered deformation of $\tilde{A}$

$\tilde{f}_\lambda = \Gamma(T_\lambda \otimes \cdot): \text{coh}(D^b_\lambda) \to \hat{\mathfrak{F}}_\lambda \text{-mod}$
3.2) Property $\heartsuit$

We will need a certain condition on $\Gamma$:

(\heartsuit) $\Gamma$ has direct summand of rank 1

By twisting $\Gamma$ with line bundle can assume this summand is $O_x \sim$ idempotent $e$ s.t.

- $e \tilde{\mathfrak{A}}_\lambda e = \mathfrak{A}_\lambda$
- $\Gamma_\lambda \cong e \tilde{\Gamma}_\lambda$

In particular, if $\Gamma_\lambda$ is equivalence $\Rightarrow \Gamma_\lambda$ is exact. Here are examples when (\heartsuit) holds (for some choices of $\Gamma$):

1) $X = T^*(G/B)$ (Bezrukavnikov-Mirkovic-Ramnyin)

2) $X$ is a symplectic resolution of symplectic quotient singularity $V/\Gamma$ (Bezrukavnikov-Kaledin). For $\Gamma$ can choose a "Procesi bundle" $\tilde{A} = \text{End}(\Gamma) = C[V] \# \Gamma$, $\tilde{\mathfrak{A}}_\lambda$ is symplectic reflection algebra.
3) $X$ is a smooth version of the Coulomb branch of a gauge theory (Webster). These include finite/affine type $A$ Nakajima quiver varieties.

**Remark:** These tilting generators are constructed starting from quantizations in characteristic $p$.

3.3) Localization for $\tilde{U}_\lambda$.

- $R\Gamma(T \otimes \cdot) : D^b(\text{Coh } X) \simto D^b(\tilde{A}^{\text{-mod}})$
- $R\Gamma(T_\lambda \otimes \cdot) : D^b(\text{Coh } D_\lambda) \simto D^b(\tilde{\mathfrak{F}}_\lambda^{\text{-mod}})$

- $\Gamma_\lambda$ is equivalence $\Rightarrow \mathfrak{F}_\lambda$ & $\tilde{\mathfrak{F}}_\lambda$ are Morita equivalent $\Rightarrow \tilde{\Gamma}_\lambda$ is equivalence.
Theorem 1 (I.1.21) Assume (for simplicity $X = T^*(G_b)$)

Let $\lambda$ be regular dominant. Then $\bar{\Gamma}_\lambda$ is an equivalence (can be generalized as long as $(\heartsuit)$ holds – plus some technicalities).

Ideas of proof:
1) Identify a locus where $\bar{\Gamma}_\lambda$ is guaranteed to be equivalence.
   (a) Deep enough inside the dominant chamber (general reasons)
   (b) Let $\Sigma$ be a hyperplane intersecting the dominant chamber and parallel to its wall, deep enough in their intersection (from reducing to the case of $T^*(P')$)

   (This works as stated for integral $\lambda$ but can be generalized by dropping "non-essential" walls).
Example (rk 2, integral λ)

\[ \Gamma_{\lambda} \text{ is equiv. for reason (b)} \]

\[ \Gamma_{\lambda} \text{ is equiv. for reason (a)} \]

\[ \tilde{\Gamma}_{\lambda} \text{ is equivalence} \]

1) To handle the missing points (in green) use "translation bimodules"

\[ \tilde{U}_{\lambda,x} \in \tilde{U}_{\lambda + \lambda} \text{-} \tilde{U}_{\lambda} \text{-bimod, } \lambda \in \Lambda \text{ (wt. lattice)}\]

Construction (approximate, the actual is more technical)

\[ x \rightarrow \mathcal{O}(x) \in \mathcal{O}(x) \rightarrow \text{quantization } \tilde{D}_{\lambda} \text{ to } \tilde{D}_{\lambda + \lambda} \text{-} \tilde{D}_{\lambda} \text{-bimodule} \]

\[ \tilde{U}_{\lambda,x} \cong \Gamma \left( \tilde{T}_{\lambda + \lambda} \otimes \tilde{D}_{\lambda,x} \otimes \tilde{D}_{\lambda} \Gamma_{\lambda}^{*} \right) \]

\[ : = \text{ for dominant } x. \]
Role translation bimodules play:

- \( \tilde{\mathcal{U}}_{\lambda, x} \otimes_{\tilde{\mathcal{U}}_{\lambda}} \tilde{\mathcal{U}}_{\lambda} : D^b(\tilde{\mathcal{U}}_{\lambda}-\text{mod}) \rightarrow D(\tilde{\mathcal{U}}_{\lambda+x}-\text{mod}) \)

- Suppose \( \tilde{\lambda} \) is an equivalence. TFAE
  - \( \tilde{\lambda}+\tilde{x} \) is equivalence
  - \( \tilde{\mathcal{U}}_{\lambda, x} \) is Morita equiv. bimodule

- for weights \( \lambda, x_1, x_2 \) have homomorphism
  \[ \tilde{\mathcal{U}}_{\lambda+x_1, x_2} \otimes_{\tilde{\mathcal{U}}_{\lambda+x_2}} \tilde{\mathcal{U}}_{\lambda, x_1} \rightarrow \tilde{\mathcal{U}}_{\lambda, x_1+x_2} \]

Under various conditions on \( \lambda, x_1, x_2 \) this is an isomorphism. Playing with these conditions & the previous two bullets leads to a proof of the theorem

\( \square \)
4) Categories O.

4.1) Cartan subquotients.

X conical symplectic resolution, $\mathcal{T}_AX$

Hamiltonian torus action w. finitely

many fixed points.

$p := H^2(X,\mathbb{C})$-parameter space for

quantizations $\sim$ universal quantizations

$D_p$ of $X$ & $\mathfrak{A}_p$ of $\mathcal{O}[X] \triangleq \mathcal{O}[\mathcal{P}]$-algebras with:

$A_p = \Gamma(D_p)$, $\mathcal{D}_\lambda = D_p \otimes \mathcal{O}[\mathcal{P}] C\lambda$, $\mathfrak{A}_\lambda = \mathfrak{A}_p \otimes \mathcal{O}[\mathcal{P}] C\lambda$.

Pick generic $\psi: \mathbb{C}^* \to T \sim$ gradings

$\mathfrak{A}_p = \bigoplus \mathbb{R}^i \mathfrak{A}_p^i$, $\mathcal{D}_\lambda = \bigoplus \mathbb{R}^i \mathcal{D}_\lambda^i$, $A_\lambda = \ldots$, $\mathcal{D}_\lambda = \ldots$

**Definition:** Cartan subquotients:

$C_p(\mathfrak{A}_p) = \mathfrak{A}_p^0 / \sum_{i > 0} \mathfrak{A}_p^i \mathfrak{A}_p^i$, $C_p(D_p)$, $C_\lambda(\mathfrak{A}_p)$, $C_\lambda(D_\lambda)$. 

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Properties: \( \cdot \mathcal{C}_f(D_p), \mathcal{C}_f(D_\lambda) \) are sheaves on \(X^T \) (finite set). In fact, 
\[
\mathcal{C}_f(D_p) = \mathcal{C}[P]^{\otimes x_T}, \quad \mathcal{C}_f(D_\lambda) = \mathcal{C}^{\otimes x_T}
\]

\( \cdot \) have natural homomorphism 
\[
\mathcal{C}_f(A_p) \to \mathcal{C}_f(D_p) \text{ induced by } A_p \cong \Gamma(D_p)
\]
\[
\cdot \quad \mathcal{C}_f(A_\lambda) = \mathcal{C}_f(A_p) \otimes \mathcal{C}[P] C_\lambda
\]

Example: \(X = T^* (G/B), \, p = f^*, \, D_p = \left( \eta^* D_{\mathbb{C}^*} \right)^T \)
where \(\eta: G/U \to G/B\)
\[
A_p = \mathcal{C} [f^*] \otimes \mathcal{C} [f^*] w U, \quad \mathcal{C}_f(A_p) = \mathcal{C} [f^*] \otimes \mathcal{C} [f^*]
\]

\(T^* (G/B)^T \to W \ni w \to \eta: f^* \xrightarrow{x \mapsto (x, w(x))} f^* f^* f^* w f^*\)

The homomorphism 
\[
\mathcal{C}_f(A_p) \to \mathcal{C}_f(D_p) = \mathcal{C} [f^*] \oplus \bigoplus_{w \in W} \mathcal{C}_w
\]
4.2) Categories $\mathcal{O}$

**Definition:** Category $\mathcal{O}_Y(\mathcal{A}_\lambda)$ consists of fin. generated $\mathcal{A}_\lambda$-modules where $\mathcal{A}_\lambda = \bigoplus_{i > 0} \mathcal{A}_i$ acts locally nilpotently.

**Lemma:** For a fin. gen. $\mathcal{A}_\lambda$-module $M$ TFAE

1) $M \in \mathcal{O}_Y(\mathcal{A}_\lambda)$

2) $M$ admits a (weakly $G$) $\mathcal{G}(\mathbb{C}^*)$-equivariant structure & is supported on contracting locus for $\gamma$ in $Y$.

**Definition:** Category $\mathcal{O}_Y(\mathcal{D}_x)$ is the full subcategory in $\text{Coh}(\mathcal{D}_x)$ consisting of all objects that admit a $\mathcal{G}(\mathbb{C}^*)$-equivariant structure & are supported on contracting locus for $\gamma$ in $X$.

**Example:** for $\gamma$ dominant $\mathcal{O}_Y(\mathcal{D}_x^\lambda) = \{ U\text{-equivariant } \mathcal{D}_x^\lambda\text{-modules} \}$.
Note that $\Gamma_\lambda$, $\text{Loc}_\lambda$ restrict to
\[ \mathcal{O}_V(D_\lambda) \overset{\sim}{\rightarrow} \mathcal{O}_V(A_\lambda) \]

**Lemma (description of simples)**
- $\text{Irr} \mathcal{O}_V(D_\lambda) \overset{\sim}{\rightarrow} X^\perp = \text{Irr} \mathcal{O}_V(D_\lambda)$
- $\text{Irr} \mathcal{O}_V(A_\lambda) \overset{\sim}{\rightarrow} \text{Irr} \mathcal{O}_V(A_\lambda)$
  can be shown to have $\dim < \infty$.

### 4.3) $O$-regular parameters & localization

**Definition:** Quantization parameter $\lambda$ is **$O$-regular** if $\mathcal{O}_V(A_\lambda) \overset{\sim}{\rightarrow} \mathcal{O}_V(D_\lambda)$
- Zariski open locus in $H^2(X, \mathbb{C})$, $\neq \emptyset$.

**Example:** For $X = T^*(G/B)$, $O$-regular $\Leftrightarrow$ regular.

**Observation:** If $\Gamma_\lambda$ is exact $\Rightarrow \Gamma_\lambda \circ \text{Loc}_\lambda \cong \text{id}$

$\Rightarrow \Gamma_\lambda : \text{Coh}(D_\lambda) \rightarrow \mathcal{A}$-mod & $\mathcal{O}_V(D_\lambda) \rightarrow \mathcal{O}_V(A_\lambda)$
are Serre quotient functors.
So: if \( \Gamma_x \) is exact & \( \lambda \) is 0-regular, then
\[
\Gamma_x: O_\lambda(D_\lambda) \simto O_\lambda(\mathfrak{h}_\lambda)
\] - Serre quotient functor between categories w. same (finite) number of simples.

Example: \( X = T^*(G/B) \), \( \lambda \) is regular (\( \Rightarrow \) O-regular) & dominant (so \( \Gamma_x \) is exact by Theorem 1)
Then \( \Gamma_x: O_\lambda(D_{\lambda/\mathfrak{h}}) \simto O_\lambda(\mathfrak{h}_\lambda) \).

The case of general \( X \) is handled using:

**Theorem 2** (I.I. 21) Let \( \lambda \in H^2(X, \mathbb{C}) \) be s.t. \( \Gamma_x \) is exact. Then in a neighborhood of \( \lambda \) (in, say, usual topology) the complement to locus of O-regular parameters has pure codim 1.