Kazhdan-Lusztig

FSB

"From quantum groups to factorizable sheaves"

Feigin-Frenkel

"screening operators satisfy quantum Serre relations"

Varchenko-Schechtman

$$
\begin{tikzcd}
\mathcal{O}_\kappa \arrow[rr, shift left, "\sim"] \arrow[drr, "KL"'] & & \text{Factorizable sheaves} \arrow[ll, shift left, "Scr"] \arrow[d, leftrightarrow, "FSB"] \\
& & \mathcal{O}_q
\end{tikzcd}
$$

A - chiral algebra ; M - chiral module

$$j_\cdot j^*(A \boxtimes M) \longrightarrow \Delta_* M$$

$$\begin{tikzcd} & X \times X \arrow[dl] \arrow[dr] & \\ X & & X \end{tikzcd}$$

$$\begin{tikzcd} & \Delta_* M \arrow[dl, "(id \otimes h)"'] \arrow[dr, "(h \otimes id)"] & \\ M & & M \end{tikzcd}$$

$$\Delta_* M = M \otimes_{O_X} D_X$$

- $(z,0)$ acts by $(m \otimes D)\xi = m \otimes D\xi$

$\cdot\ (0, \xi)$ act as $m\xi \otimes D - m \otimes \xi D$.

$$M^{\ell} \otimes D_x \xrightarrow{\partial} M \otimes D_x \xrightarrow{(id \otimes h)} M$$

$m \in \Gamma(X, M)$ s.t.

$$\begin{tikzcd}
A \boxtimes M \arrow[r] \arrow[rr, bend right, "{[\,,\,]}"'] & j_*j^*(A \boxtimes M) \arrow[r, "{\{\,,\,\}}"] & \Delta_* M
\end{tikzcd}$$

$\forall a \in A$

$$[a, m] = \partial \Phi(a)$$

$$\Phi : A \longrightarrow M^{\ell} \otimes D_x$$

1) $j_* j^*(\mathcal{M} \boxtimes \mathcal{M}_1) \longrightarrow \Delta_* \mathcal{M}_2$

$m \in \Gamma(X, \mathcal{M})$ is a screening.

$$\begin{tikzcd} \mathcal{M} \boxtimes \mathcal{M}_1 \arrow[r] & j_* j^*(\mathcal{M} \boxtimes \mathcal{M}_1) \arrow[r] & \Delta_* \mathcal{M}_2 \arrow[d, "(\text{h}\otimes\text{id})"'] \\ & & \mathcal{M}_2 \end{tikzcd}$$

$m \boxtimes -$

$$\mathcal{M}_1 \xrightarrow{[m,-]} \mathcal{M}_2 .$$

__Lemma__ m screening $\Longrightarrow$ $[m,-]$ is a map of chiral A-modules

2) $\mathcal{M}_1 \quad \mathcal{M}_n$ at points $x_1 \ldots x_n \in X$

$H_\nabla(A, \mathcal{M}_1 \quad \mathcal{M}_n)$

$$C^{\bullet}_{dR}(X_x, H_{\bullet}(A, M \otimes M_1 \dots M_n))$$

$$\cong$$

$$D\mathcal{M}(X - x_1, \dots x_n)$$

3)
$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & \tilde{M} \arrow[r] & A \arrow[r] & 0 \\
& & & \omega_x \arrow[u] & \\
& & & \omega_x \otimes D_x \arrow[u] \arrow[uul, dashed] &
\end{tikzcd}$$

+ rigidification

<u>Examples</u>

$A_{\mathfrak{g},k}$

$\mathfrak{g}$ - reductive Lie algebra
k - level

$$\mathfrak{g} \otimes D_x \rightsquigarrow A_{\mathfrak{g},k}$$

$$(\mathfrak{g}\otimes\mathcal{O}_x) \overset{T}{\times} \mathcal{S}(\omega_x) \rightsquigarrow \mathcal{A}_{\mathfrak{g},\kappa}$$

ω_x - canonical line bundle on X.

$\mathfrak{g} = sl_2$

$$M = \mathbb{M}_\kappa^{-\alpha}$$

$$0 \to M^{-\alpha} \to M^0 \to k \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{M}^{-\alpha} \arrow[r] & \mathbb{M}^0 \arrow[r] & \mathbb{V}^0 \arrow[r] \arrow[d, equal] & 0 \\
 & & & \mathcal{A}_{\mathfrak{g},\kappa} &
\end{tikzcd}$$

screening element in $\mathbb{M}^{-\alpha}$

$$\omega_x^{-1} \xleftrightarrow{h^{-\alpha}} \mathbb{M}_\kappa^{-\alpha}$$

$$m = e\cdot h^{-\alpha}$$

$$m_i \in \mathbb{M}_\kappa^{-\alpha_i} \longrightarrow \mathbb{W}\mathrm{ak}_\kappa^{-\alpha_i}.$$

$$\mathcal{O}_\kappa := \hat{\mathfrak{g}}\text{-}\mathrm{mod}_\kappa^{I}$$

$$KL(G)_\kappa := \hat{\mathfrak{g}}\text{-}\mathrm{mod}_\kappa^{G(\mathcal{O})}$$

$$KL(T)_\kappa := \hat{\mathfrak{t}}\text{-}\mathrm{mod}_\kappa^{T(\mathcal{O})}.$$

$$\mathrm{BRST} : KL(G)_\kappa \longrightarrow KL(T)_\kappa$$

$$\mathcal{O}_\kappa \longrightarrow KL(T)_\kappa$$

$$\Omega_\kappa^{\mathrm{Heis}} := \mathrm{BRST}(\mathcal{A}_{\mathfrak{g}^\vee,\kappa})$$

$$\cap$$

$$\mathrm{Alg}^{\mathrm{Fact}}(KL(T)_\kappa)$$

$$\mathrm{BRST}^{\mathrm{enh}} : \mathcal{O}_\kappa \overset{\dashleftarrow}{\longrightarrow} \Omega^{\mathrm{Heis}}_{\kappa\text{-}\mathrm{Lat}}{}^{\mathrm{Fact}}(\mathrm{KL}(T_\kappa))$$

<u>Thm</u> (Lin Chen, Charles Fu)

$\mathrm{BRST}^{\mathrm{enh}}$ is an equivalence.

(for κ irrational or positive)

In this talk will need

κ irrational or negative $\rightsquigarrow$

renormalized version of the right-hand side.

$$A \in \mathrm{Alg}^{\mathrm{Fact}}(\mathrm{KL}(T)_\kappa)$$

its supported on Λ^{neg}

A^0 is $\mathcal{A}_{T,\kappa}$.

$(A = \Omega^{\mathrm{Heis}}_\kappa)$.

$$\mathrm{Conf} = \bigsqcup_{\lambda \in \Lambda^{neg}} \mathrm{Conf}^\lambda$$

$$\mathrm{Conf}^\lambda = \prod_{i \in I} X^{(n_i)}$$

$$\lambda = \sum -n_i \alpha_i$$

$\kappa \rightsquigarrow$ gerbe on Conf.

$A \longleftrightarrow$ ~~G~~ Gerbe-twisted factorizable D-modules on Conf.

Apply Riemann-Hilbert on FSB and obtain $\mathcal{D}_q$.

Ω^{Heis} is an object of the KLCThm

on $\mathrm{Conf} = \bigcup \mathrm{Conf}^\lambda$

$$\mathrm{Conf}^\lambda - \Delta X \xrightarrow{j^\lambda} \mathrm{Conf}^\lambda$$

κ - irrational.

$$\Omega^{\mathrm{Hess}}\big|_{\mathrm{Conf}^\lambda} = j^\lambda_{!*}\, j^{\lambda *}(\Omega^{\mathrm{Hess}})$$

1) $\lambda \neq w(\rho) - \rho$, $w \in W$ $\Rightarrow$

$$j^\lambda_! \xrightarrow{\sim} j^\lambda_{!*} \xrightarrow{\sim} j^\lambda_*$$

2) $\lambda = w(\rho) - \rho$, $\ell(w) \geq 3$.

$$j^\lambda_{!*} = H^0(j_*)$$

3) $\lambda = w(\rho) - \rho$ $\quad \ell(w) = 2$

$$0 \to j^\lambda_{!*} \longrightarrow j^\lambda_* \longrightarrow \Delta_*(\mathcal{O}_X) \to 0$$

expresses the quantum Serre relations.

BRST : $\mathcal{A}_{\check{\mathfrak{g}},\kappa\text{-mod}} \longrightarrow \mathcal{A}_{t,\kappa}$

Wak : $\mathcal{A}_{t,g} \longrightarrow \mathcal{A}_{\check{\mathfrak{g}},\kappa}$.

$$\Omega^{Wak} := Wak(\Omega^{Heis}) \in Alg_g^{Fact}(\mathcal{A}_{\check{g}\text{-mod}}).$$

$$\begin{tikzcd}
\Omega^{Heis}\text{-mod}^{Fact}(KL(T_\kappa)) \arrow[r] \arrow[dr, dashed, "\text{desired functor.}"'] & \Omega^{Wak}\text{-mod}^{Fact}(\mathcal{A}_{\check{g}\text{-mod}}) \arrow[d, "Ser"] \\
& \mathcal{A}_\kappa
\end{tikzcd}$$

$M \in \mathcal{A}\text{-mod}(X) \rightsquigarrow$

$m \in \Gamma(X, M)$ screening section.

This action can be generalized

$m \in \Gamma(X^n, M)$

$\mathcal{M} \in \mathcal{A}\text{-mod}^{ch}(X^n)$.

Makes sense to talk about screening section of

$\mathcal{A}_{gln}^{v\text{-mod}}(\mathrm{Conf}^\lambda)$.

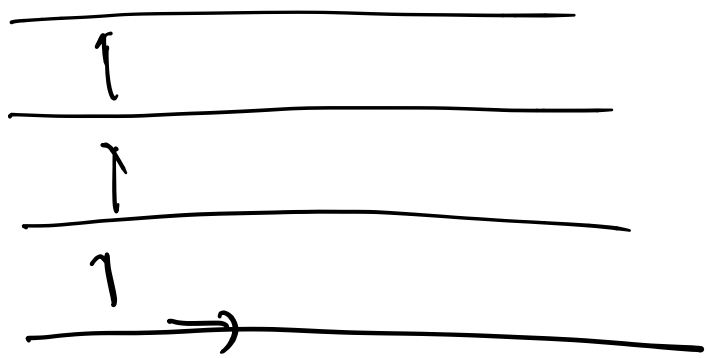
$M_i \in \mathrm{Whit}_n^{-\alpha_i} \longrightarrow$

$\mathring{m}^\lambda$ - screening section of

Ω^{Whit} on $\mathring{\mathrm{Conf}}^\lambda$

<u>Thm</u> $\mathring{m}^\lambda$ extend (uniquely) to a section m^λ of Ω^{Whit} on Conf^λ.

$M^\cdot$ be a $\mathbb{Z}$-gr. graded complex of vector spaces



∂ : vertical differential
(of wt 1 and degree 1) $\longrightarrow$

M_t over $k[t]$, equivariant wrt $\mathbb{G}_m$.

$$M_t / t M_t \simeq M^\cdot$$

$$\widetilde{M} = M_t / (t-1) M_t .$$

- The weights are non-negative
- $\forall j$ M^i will be accepted in filtr. $\geq j$

for $i \gg 0$;

$A^\cdot$ graded associative algebra.

$$A^\cdot \longrightarrow A_t \longrightarrow \widetilde{A}.$$

$$k \cdot \xi \longrightarrow A$$

DG Lie algebra.

$$k[\xi] \longrightarrow A$$

associative algebra.

$$A_0 \longrightarrow A$$

$$A_0[\xi] \longrightarrow A$$

$$M^\cdot \rightsquigarrow M_t \longrightarrow \widetilde{M}$$

will be A_0-modules.

$\mathcal{A}$ - factorization algebra.

$$\begin{array}{c}\mathcal{A}\\ \uparrow\\ \mathcal{A}_0\end{array}$$

Maurer-Cartan data:

$$\xi^n \in \operatorname{Ext}^{2n}\left(\mathcal{A}_0^{(n)}, \mathcal{A}^{(n)}\right)$$

$\mathcal{A}_0$-~~mod~~ ${}^{Fact}(X^n)$. $\rightarrow$

deform dual $\mathcal{A}$-modules as $\mathcal{A}_0$-modules.

$$\mathcal{A}_0 = \mathcal{A}_{g,\kappa}.$$

$$\mathcal{A} = \Omega^{\mathrm{ch}}.$$

the datum of S^λ is exactly the Maurer-Cartan datum.

$\Omega^{\text{Hen}}_{\text{-nil}}$