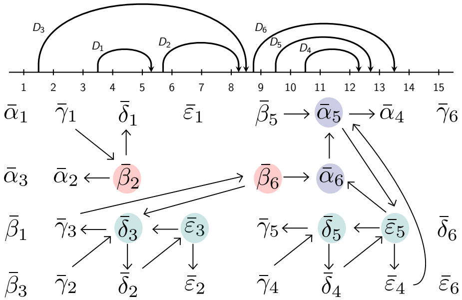
Cluster algebras and tilings for the $m = 4$ amplituhedron

Lauren K. Williams, Harvard



Based on joint work with:
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Thanks to: National Science Foundation

Overview of the talk

- Part 1: The positive Grassmannian, the amplituhedron, and the BCFW tiling conjecture

- Part 2: Cluster algebras, and the cluster adjacency conjecture for the amplituhedron

Review of the Grassmannian

The **Grassmannian** $Gr_{k,n}(\mathbb{C}) := \{V \mid V \subset \mathbb{C}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}$ by a full-rank $k \times n$ matrix C.

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $\langle I \rangle_C$ is the minor of the $k \times k$ submatrix of C in column set I.

What is the positive Grassmannian?

Background: Lusztig's total positivity for G/P 1994, Rietsch 1997, Postnikov's 2006 preprint on the *totally non-negative* (TNN) or "positive" Grassmannian.

Let $Gr_{k,n}^{\geq 0}$ be subset of $Gr_{k,n}(\mathbb{R})$ where Plucker coords $\langle I \rangle \geq 0$ for all I.

Inspired by matroid stratification, one can partition $Gr_{k,n}^{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}}^{tnn} := \{C \in Gr_{k,n}^{\geq 0} \mid \langle I \rangle_C > 0 \text{ iff } I \in \mathcal{M}\}$.

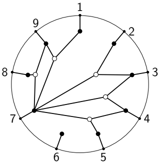
In contrast to terrible topology of matroid strata ...

(Postnikov) If $S_{\mathcal{M}}^{tnn}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. So we have *positroid cell decomposition*

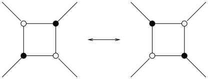
$$Gr_{k,n}^{\geq 0} = \sqcup S_{\mathcal{M}}^{tnn}.$$

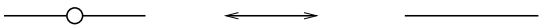
Cells of the positive Grassmannian

Thm (Postnikov): The positroid cells of $Gr_{k,n}^{\geq 0}$ are in bijection with equivalence classes of *planar bicolored (plabic) graphs.*



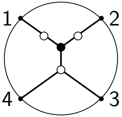
A *plabic graph* is a planar graph embedded in a disk, with boundary vertices labeled $1, 2, \ldots, n$, and internal vertices colored black or white. Say two plabic graphs *move-equivalent* if they can be obtained from each other by a series of local moves:

(M1):  (M2): 

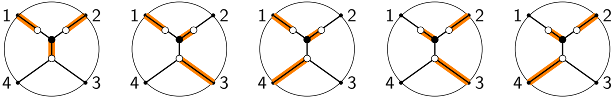
(M3): 

How to read off a positroid cell from a plabic graph

- Positroid cells $\leftrightarrow$ move-equivalence classes of *plabic graphs*



- Using moves we can assume graph G is bipartite and that every boundary vertex is incident to a white vertex.
- Let $\mathcal{M}(G) := \{\partial(P) \mid P$ is an **almost perfect matching** of $G\}$.



E.g. for graph above, get $\mathcal{M}(G) = \{12, 13, 14, 23, 24\}$. So this represents a positroid cell of $\mathrm{Gr}_{2,4}$ in which precisely these Plücker coordinates are positive.

- Theorem (Postnikov): $\mathcal{M}(G)$ is the set of nonzero Plücker coordinates of a positroid cell, and all cells obtained this way.

What is the amplituhedron?

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$, Arkani-Hamed–Trnka (2013).

Fix n, k, m with $k + m \leq n$.
Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ be an $n \times (k+m)$ matrix with max'l minors positive.
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to span(CZ).
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Special cases:

- If $m = n - k$, $\mathcal{A}_{n,k,m} = Gr_{k,n}^{\geq 0}$.

What is the amplituhedron?

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ (max minors > 0).
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Special cases:

- If $k = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{1,1+m}$ is equivalent to a cyclic polytope with n vertices in $\mathbb{P}^m$:
  E.g. if $m = 2$, let $Z_1, \dots, Z_n$ denote rows of $Z \in \mathrm{Mat}_{n,3}^{>0}$.
  Positivity implies they represent vertices of convex polytope in $\mathbb{P}^2$.
  Image of $Gr_{1,3}^{\geq 0}$ under $\tilde{Z}$ gives entire polytope.

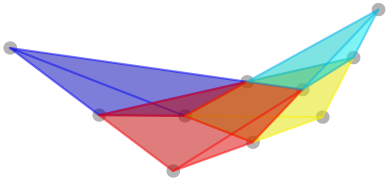
What is the amplituhedron?

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ (max minors > 0).
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Special cases:

- If $m = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{k,k+1}$ is homeomorphic to the bounded complex of the cyclic hyperplane arrangement (Karp–W.).
  E.g. $\mathcal{A}_{5,3,1}$:



Remarks on the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ (max minors > 0).
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}(Z)$ is a full-dimensional (km-dimensional) subset of $Gr_{k,k+m}$.
- Clearly $\mathcal{A}_{n,k,m}(Z)$ depends on the choice of matrix Z. However, as we'll see, many properties of $\mathcal{A}_{n,k,m}(Z)$ do not depend on Z.
- In order to talk about $\mathcal{A}_{n,k,m}(Z)$ as a subset of $Gr_{k,k+m}$, we need to have some good coordinates on $Gr_{k,k+m}$ which take Z into account!

Coordinates for the amplituhedron

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}_{n,k+m}^{>0}$ with rows $Z_1, \dots, Z_n$.
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Let $Y \in \mathcal{A}_{n,k,m}(Z) \subset Gr_{k,k+m}$ (viewed as matrix).
Given $I = \{i_1 < \dots < i_m\} \subset [n]$, let

$$\langle\langle I \rangle\rangle = \langle\langle YZ_I \rangle\rangle = \langle\langle YZ_{i_1} \dots Z_{i_m} \rangle\rangle := \det \begin{bmatrix} - & Y & - \\ - & Z_{i_1} & - \\ & \vdots & \\ - & Z_{i_m} & - \end{bmatrix}$$

Call it *twistor coordinate* $\langle\langle YZ_I \rangle\rangle$ (Arkani-Hamed–Thomas–Trnka).
Rk: Y is determined by twistor coords; the twistor coordinate $\langle\langle YZ_I \rangle\rangle$ equals the Plücker coordinate $\langle I \rangle_{Y^\perp Z^t}$.
We refer to a polynomial in twistor coordinates as a *functionary*.

Motivation for the amplituhedron?

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ (max minors > 0).
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Motivation for the $m = 4$ amplituhedron ($\mathcal{N} = 4$ SYM):

- the recurrence of Britto–Cachazo–Feng–Witten (2005) expresses scattering amplitudes as sums of rat'l functions of momenta. Indiv terms have "spurious poles" – singularities not present in amplitude.
- Hodges (2009) observed that in some cases, the amplitude is the volume of a polytope, with spurious poles arising from internal boundaries of a triangulation of the polytope. Asked if in general each amplitude is the volume of some geometric object.
- AH–T found the amplituhedron as the answer to this question; BCFW recurrence is interpreted as "triangulation" of $\mathcal{A}_{n,k,4}(Z)$.

Tiles and tilings of the amplituhedron

Have $Gr_{k,n}^{\geq 0} = \sqcup_\pi S_\pi$ cell complex, and $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ a continuous surjective map onto km-dim'l amplituhedron $\mathcal{A}_{n,k,m}(Z)$.

If $\tilde{Z}$ is injective on a km-dim'l cell S_π, we say that $Z_\pi := \overline{\tilde{Z}(S_\pi)}$ is a *tile* for $\mathcal{A}_{n,k,m}(Z)$. A $\tilde{Z}$*-induced tiling* (or positroid tiling) of $\mathcal{A}_{n,k,m}(Z)$ is a collection $\{Z_\pi \mid \pi \in \mathcal{C}\}$ of tiles, such that:

- their union equals $\mathcal{A}_{n,k,m}(Z)$
- their interiors are pairwise disjoint

Tilings of the amplituhedron when $k = 1, m = 2, n = 5$

The map $\tilde{Z} : Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ becomes $\tilde{Z} : Gr_{1,5}^{\geq 0} \to Gr_{1,3}$.

BCFW tiling conjecture

Have $Gr_{k,n}^{\geq 0} = \sqcup_\pi S_\pi$ cell complex, and $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ a continuous surjective map onto km-dim'l amplituhedron $\mathcal{A}_{n,k,m}(Z)$.

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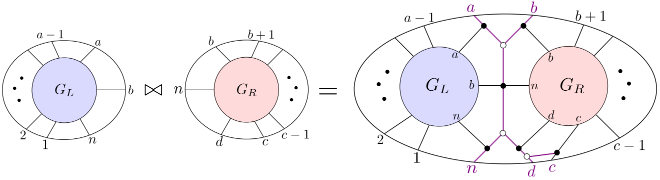
- their union equals $\mathcal{A}_{n,k,m}(Z)$
- their interiors are pairwise disjoint

The BCFW tiling conjecture:
Arkani-Hamed–Trnka interpreted each way of iterating the BCFW recurrence as giving a collection of $4k$-dimensional cells in $Gr_{k,n}^{\geq 0}$ whose images conjecturally tile the $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$.

General BCFW cells

For $1 \leq a < b < \cdots < c < d < n$, if we have two plabic graphs on $\{1, 2, \ldots, a, b, n\}$ and $\{b, \ldots, c, d, n\}$, we can build their *BCFW product*:

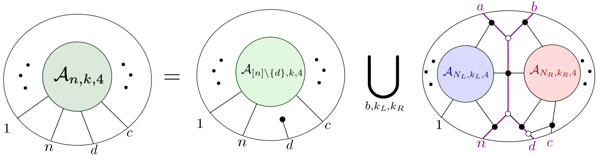


The set of *general BCFW cells* is defined recursively:

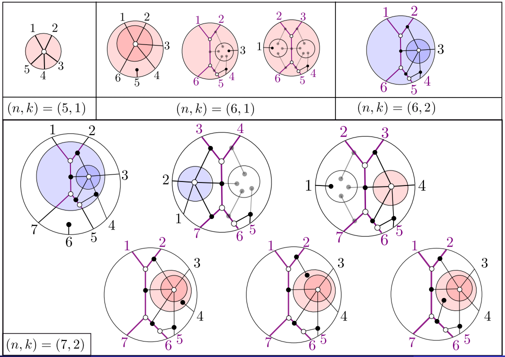
1. For $k = 0$, the trivial cell $\mathrm{Gr}_{0,n}^{>0}$ is a general BCFW cell.
2. If S is a general BCFW cell, then so is any cell obtained by inserting a zero column, performing a cyclic shift, or reflecting it.
3. If S_L and S_R are general BCFW cells on N_L and N_R, then so is their BCFW product $S_L \bowtie S_R$.

Standard BCFW cells: the special case where we disallow cyclic shifts/ reflections, and we only insert a zero column in the penultimate position.

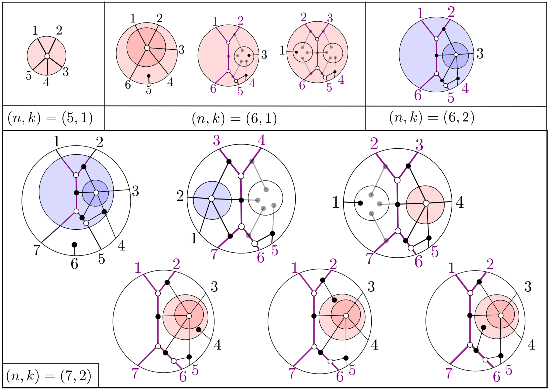
The BCFW recurrence and BCFW tiling conjecture



BCFW tiling conjecture: the above recurrence produces a collection of BCFW cells whose images tile the amplituhedron.



The BCFW recurrence and the BCFW tiling conjecture



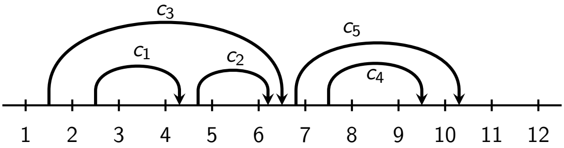
Note: When we iterate the BCFW recurrence, we always arrive at a collection of $N(n-3, k+1) = \frac{1}{k+1}\binom{n-4}{k}\binom{n-3}{k}$ cells. (Narayana number!)

Standard BCFW cells and combinatorics

Recall that standard BCFW cells are the ones obtained by iterating the BCFW recurrence in a canonical way; enumerated by Narayana numbers.

$\exists$ explicit bijections between the standard BCFW cells in $Gr_{k,n}^{\geq 0}$ and:

- pairs of noncrossing lattice paths in a $k \times (n-k-4)$ rectangle, equivalently, plane partitions in a $2 \times k \times (n-k-4)$ rectangle (Karp-W.-Zhang)
- chord diagrams (Evan-Zohar–Lakrec–Tessler)



Theorem (Evan-Zohar–Lakrec–Tessler)

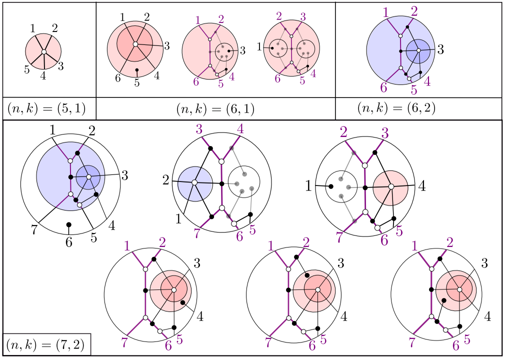
The BCFW tiling conjecture holds for the standard BCFW cells.

Proof used explicit coordinates for cells coming from chord diagrams.

BCFW tiling Theorem for general BCFW cells (EZ-L-P-SB-T-W)

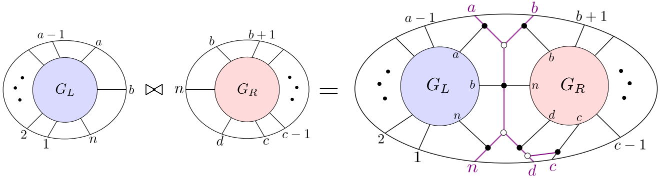
Every collection $\{S_{\mathfrak{r}}\}$ of general BCFW cells obtained by iterating the BCFW recurrence $\rightsquigarrow$ tiling $\{Z_{\mathfrak{r}}\}$ of the amplituhedron $\mathcal{A}_{n,k,4}(Z)$. That is:

- the amplituhedron map is injective on each general BCFW cell $S_{\mathfrak{r}}$, i.e. $Z_{\mathfrak{r}} := \overline{\tilde{Z}(S_{\mathfrak{r}})}$ is a *tile*;
- the open tiles $\{Z_{\mathfrak{r}}^{\circ}\}$ are pairwise disjoint;
- and the tiles in $\{Z_{\mathfrak{r}}\}$ cover the amplituhedron $\mathcal{A}_{n,k,4}(Z)$.



Ideas behind the proof

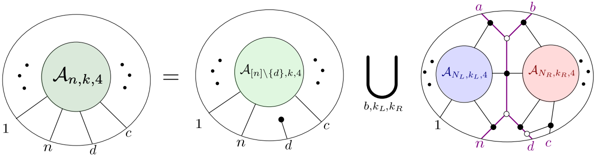
- We have a recursive structure on BCFW cells: they are built up using operations of cyclic shift, reflection, and the BCFW product.



- To show that $\tilde{Z}$ is injective on a BCFW cell, we show how to invert the map. We construct a particular parameterization of the elements of each BCFW cell $S_{\mathfrak{r}}$ compatible with the above operations.
- We inductively define some *coordinate functionaries*, functions on the image $\tilde{Z}(S_{\mathfrak{r}}) \subset Gr_{k,k+4}$, and show that $\tilde{Z}(S_{\mathfrak{r}})$ is the subset of $Gr_{k,k+4}$ where the coordinate functionaries are positive. We can then use the coordinate functionaries to explicitly invert $\tilde{Z}$ on the image.

Ideas behind the proof

- We have a recursive structure for the *BCFW collections* $\{S_\mathfrak{r}\}$ of cells whose images $\{Z_\mathfrak{r}\}$ are supposed to tile the amplituhedron:



- To show that any two elements $Z_\mathfrak{r}$ and $Z_{\mathfrak{r}'}$ of $\{Z_\mathfrak{r}\}$ have disjoint interiors, we use certain collections of functionaries on $\mathcal{A}_{n,k,4}(Z)$ and show that their sign patterns are different on $Z_\mathfrak{r}$ and $Z_{\mathfrak{r}'}$. These functionaries are inductively constructed using cyclic shift, reflection, and a map called *product promotion*, so we analyze how signs of functionaries evolve when we apply these operations.
- To show that $\{Z_\mathfrak{r}\}$ covers the amplituhedron, we use the inductive construction of BCFW collections plus some results of Even-Zohar–Lakrec–Tessler and Bao-He.

Part II: The amplituhedron and cluster algebras

Is there a connection between the amplituhedron and cluster algebras?

Clue that answer should be yes:
Physicists had observed that when one calculates scattering amplitudes as rat'l functions of momenta, the poles arising in expressions seemed to be related to compatible collections of cluster variables ("cluster adjacency") – Drummond–Foster–Gurdogan, Lukowski–Parisi–Spradlin–Volovich

Next ... review notion of cluster algebra

What is a cluster algebra?

Cluster algebras (Fomin–Zelevinsky)

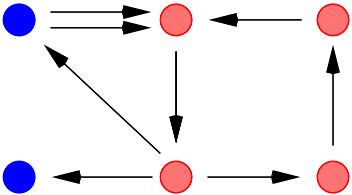
Cluster algebras are a class of commutative rings with remarkable combinatorial structure. They come with distinguished generators called *cluster variables*, and relations are encoded by *quivers* and *quiver mutation*.

Cluster varieties are varieties whose coordinate rings are cluster algebras; they come with many nice torus charts.

Examples: Grassmannians, flag varieties, Schubert varieties, ...

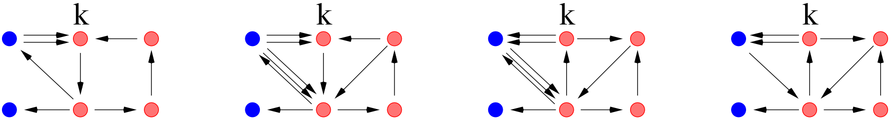
It's useful to exhibit a cluster structure because of the many general results about them (Laurent phenomenon, positivity theorem, etc).

Quivers



A *quiver* is a finite directed graph.
Multiple edges are allowed.
Oriented cycles of length 1 or 2 are forbidden.
Two types of vertices: "frozen" and "mutable."
Ignore edges connecting frozen vertices. Let s be the total number of vertices, of which $r \leq s$ are mutable.

Quiver Mutation



Let k be a mutable vertex of Q.

Quiver mutation $\mu_k : Q \mapsto Q'$ is computed in 3 steps:

1. For each instance of $j \to k \to \ell$, introduce an edge $j \to \ell$.
2. Reverse the direction of all edges incident to k.
3. Remove oriented 2-cycles.

Mutation is an involution, i.e. $\mu_k^2(Q) = Q$ for each vertex k.

Two quivers are *mutation-equivalent* if one can get between them via a sequence of mutations.

Seeds

Let $\mathcal{F}$ be a field of rational functions in s independent variables over $\mathbb{C}$. A *seed* in $\mathcal{F}$ is a pair (Q, x) consisting of:

- a quiver Q with r mutable vertices and $s - r$ frozen vertices.
- an *extended cluster* x, an s-tuple of algebraically independent (over $\mathbb{C}$) elements of $\mathcal{F}$, indexed by the vertices of Q.

frozen variables $\leftrightarrow$ frozen vertices

cluster variables $\leftrightarrow$ mutable vertices

Cluster $=$ {cluster variables }

Extended Cluster $=$ {cluster variables, frozen variables}

Seed mutation

Let k be a mutable vertex in Q and let x_k be the corresponding cluster variable. Then the seed mutation $\mu_k : (Q, x) \mapsto (Q', x')$ is defined by

- $Q' = \mu_k(Q)$
- $x' = x \cup \{x'_k\} \setminus \{x_k\}$, where

$$x_k x'_k = \prod_{j \leftarrow k} x_j + \prod_{j \rightarrow k} x_j \text{ (is the } \textit{exchange relation}\text{)}$$

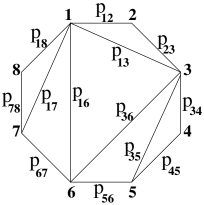
Remark: Mutation is an involution.

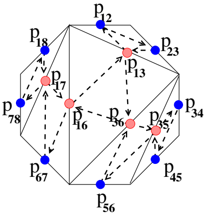
Definition of cluster algebra

- Let (Q, x) be a seed in $\mathcal{F}$, with r initial cluster variables $\{x_1, \ldots, x_r\}$ and $s - r$ frozen variables $\{x_{r+1}, \ldots, x_s\}$.
- Let χ be the (possibly infinite) set of all cluster variables, obtained by performing all possible mutation sequences starting from the initial cluster.
- Let the *ground ring* be $\mathcal{R} = \mathbb{C}[x_{r+1}^{\pm}, \ldots, x_s^{\pm}]$, the Laurent polynomial ring generated by frozen variables.
  (Alternatively let $\mathcal{R} = \mathbb{C}[x_{r+1}, \ldots, x_s]$.)
- The *cluster algebra* $\mathcal{A}(Q) := \mathcal{R}[\chi] \subset \mathcal{F}$ is the $\mathcal{R}$-subalgebra generated by χ.

Example of a cluster algebra

Label the vertices of a polygon by $1 \ldots n$, fix a triangulation, and label sides/diagonals by Plücker coordinates.
Associate a quiver, with frozen/mutable vertices at sides/diagonals, and arrows (dotted) inscribed in triangles of triangulation.



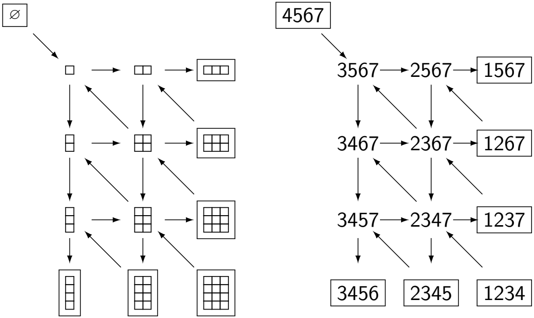


Flips of the triangulation $\leftrightarrow$ mutation $\leftrightarrow$ 3-term Plücker relations

This identifies our cluster algebra with the coordinate ring of the Grassmannian $\mathbb{C}[Gr_{2,n}]$!

Example of a cluster algebra

More generally, the coordinate ring of the Grassmannian $\mathbb{C}[Gr_{k,n}]$ has the structure of a cluster algebra (Scott). An initial seed is the following:



Note: All Plücker coordinates are cluster variables of $\mathbb{C}[Gr_{k,n}]$ but in general there are infinitely many cluster variables.

A classification of cluster variables/clusters is not understood in general. Even for $Gr_{3,n}$ there is only conjectural classification of cluster variables.

Recall: Amplituhedron and twistor coordinates

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ with rows $Z_1, \dots, Z_n$.
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Let $Y \in \mathcal{A}_{n,k,m} \subset Gr_{k,k+m}$. Given $I \in \binom{[n]}{m}$, define *twistor coordinate*

$$\langle\langle I \rangle\rangle = \langle\langle YZ_I \rangle\rangle = \langle\langle YZ_{i_1} \dots Z_{i_m} \rangle\rangle := \det \begin{bmatrix} - & Y & - \\ - & Z_{i_1} & - \\ & \vdots & \\ - & Z_{i_m} & - \end{bmatrix}$$

Twistor coordinates $\langle\langle I \rangle\rangle$ for $\mathcal{A}_{n,k,m}(Z)$ indexed by $\binom{[n]}{m}$, *just like* the Plücker coordinates $\langle I \rangle$ of $Gr_{m,n}$.
We refer to polynomials in twistor coordinates as *functionaries*.
These are functions on the amplituhedron.

The cluster adjacency conjecture for the amplituhedron

Let $Z_{\mathfrak{r}} = \overline{\tilde{Z}(S_{\mathfrak{r}})}$ be a tile of $\mathcal{A}_{n,k,m}(Z)$. We say that $Z_{\mathfrak{r}'}$ is a *facet* of $Z_{\mathfrak{r}}$ if

- $Z_{\mathfrak{r}'} \subset \partial Z_{\mathfrak{r}}$;
- cell $S_{\mathfrak{r}'}$ is contained in $\overline{S_{\mathfrak{r}}}$;
- $Z_{\mathfrak{r}'}$ has codimension 1 in $Z_{\mathfrak{r}}$.

Cluster adjacency conjecture for tiles

Let $Z_{\mathfrak{r}}$ be a tile of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$. Then for each facet $Z'_{\mathfrak{r}}$ of $Z_{\mathfrak{r}}$, there is a functionary $F_{\mathfrak{r}'}(\langle\langle I \rangle\rangle)$ which vanishes on $Z_{\mathfrak{r}'}$, such that the collection

$$\mathcal{F} = \{F_{\mathfrak{r}'}(\langle I \rangle) : Z_{\mathfrak{r}'} \text{ a facet of } Z_{\mathfrak{r}}\}$$

is a collection of compatible cluster variables for $\mathrm{Gr}_{m,n}$.

The above statement generalizes a statement for $m = 2$ which was conjectured by Lukowski–Parisi–Spradlin–Volovich (2019), and proved by Parisi-Sherman-Bennett-W (2021).

Cluster adjacency for the amplituhedron

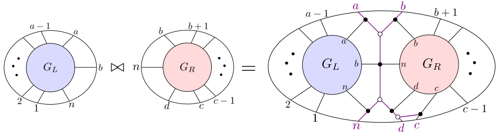
Cluster adjacency theorem for BCFW tiles (EZ-L-P-SB-T-W)

Let $Z_{\mathfrak{r}}$ be a general BCFW tile of $\mathcal{A}_{n,k,4}(Z)$. Then for each facet $Z_{\mathfrak{r}'}$ of $Z_{\mathfrak{r}}$, there is a functionary $F_{\mathfrak{r}'}(\langle\langle I \rangle\rangle)$ which vanishes on $Z_{\mathfrak{r}'}$, such that the set

$$\{F_{\mathfrak{r}'}(\langle I \rangle) : Z_{\mathfrak{r}'} \text{ a facet of } Z_{\mathfrak{r}}\}$$

is a collection of compatible cluster variables for $\mathrm{Gr}_{4,n}$.
Moreover, each such $F_{\mathfrak{r}'}$ has a fixed sign on the interior $Z_{\mathfrak{r}}^{\circ}$ of the tile.

What are these functionaries/ clust variables & how are they constructed?
Recall: the BCFW tiles are constructed recursively



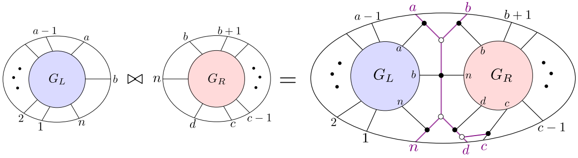
There is an algebraic counterpart of the BCFW product.

Product promotion

Choose $1 \leq a < b < c < d \leq n$ s.t. $a < b$ and $c < d < n$ are consecutive. Let $A = \widehat{\mathrm{Gr}}_{4,\{n12\ldots ab\}}$, $B = \widehat{\mathrm{Gr}}_{4,\{b\ldots cdn\}}$, and $\widehat{\mathrm{Gr}}_{4,n}$ be Grassmannians of 4-planes in vector spaces with bases labeled by $\{n, 1, 2, \ldots, a, b\}$, etc. Given a matrix $(v_1|\ldots|v_n)$ with column vectors $v_1, \ldots, v_n$, identify its Plücker coordinate $\langle i_1, \ldots, i_k\rangle$ with the element $v_{i_1} \wedge \cdots \wedge v_{i_k}$.
Then *product promotion* is the homomorphism
$\Psi : \mathbb{C}(A) \times \mathbb{C}(B) \to \mathbb{C}(\widehat{\mathrm{Gr}}_{4,n})$ induced by the following substitution:

$$b \mapsto b - \frac{\langle b\, c\, d\, n\rangle}{\langle a\, c\, d\, n\rangle}\, a \text{ on } A$$

$$n \mapsto n - \frac{\langle a\, b\, c\, n\rangle}{\langle a\, b\, c\, d\rangle}\, d + \frac{\langle a\, b\, d\, n\rangle}{\langle a\, b\, c\, d\rangle}\, c \quad \text{and} \quad d \mapsto d - \frac{\langle a\, b\, d\, n\rangle}{\langle a\, b\, c\, n\rangle}\, c \text{ on } B$$



Product promotion

Choose $1 \leq a < b < c < d \leq n$ s.t. $a < b$ and $c < d < n$ are consecutive. Let $A = \widehat{\mathrm{Gr}}_{4,\{n12\ldots ab\}}$, $B = \widehat{\mathrm{Gr}}_{4,\{b\ldots cdn\}}$, and $\widehat{\mathrm{Gr}}_{4,n}$ be Grassmannians of 4-planes in vector spaces with bases labeled by $\{n, 1, 2, \ldots, a, b\}$, etc. Given a matrix $(v_1|\ldots|v_n)$ with column vectors $v_1, \ldots, v_n$, identify its Plücker coordinate $\langle i_1, \ldots, i_k\rangle$ with the element $v_{i_1} \wedge \cdots \wedge v_{i_k}$.
Then *product promotion* is the homomorphism $\Psi : \mathbb{C}(A) \times \mathbb{C}(B) \to \mathbb{C}(\widehat{\mathrm{Gr}}_{4,n})$ induced by the following substitution:

$$b \;\mapsto\; b - \frac{\langle b\,c\,d\,n\rangle}{\langle a\,c\,d\,n\rangle}\, a \text{ on } A$$

$$n \;\mapsto\; n - \frac{\langle a\,b\,c\,n\rangle}{\langle a\,b\,c\,d\rangle}\, d + \frac{\langle a\,b\,d\,n\rangle}{\langle a\,b\,c\,d\rangle}\, c \qquad \text{and} \qquad d \;\mapsto\; d - \frac{\langle a\,b\,d\,n\rangle}{\langle a\,b\,c\,n\rangle}\, c \text{ on } B$$

Theorem (EZ-L-P-SB-T-W)

Product promotion is a *cluster quasi-homomorphism*. In particular, it takes cluster variables to cluster variables and compatible cluster variables to compatible cluster variables (up to Laurent monomial in frozens).

Cluster algebra quasi-homomorphisms

- A cluster algebra *quasi-homomorphism* (Chris Fraser) is an algebra homomorphism taking each cluster variable/ cluster to a cluster variable/cluster, up to a Laurent monomial in frozen variables.

- Requiring that cluster variables map to cluster variables is too strong.

- For elements $x, y \in \mathscr{A}$, say that x *is proportional to* y, writing $x \propto y$, if $x = My$ for some Laurent monomial M in the frozen variables.

Cluster quasi-homomorphism (can ignore!)

Given a seed $\Sigma = (Q, (x_1, \ldots, x_s))$ the *exchange ratio* of x_i (with respect to Σ) is

$$\hat{y}_\Sigma(x_i) = \frac{\prod_{j:i\to j} x_j^{\#\,\mathrm{arr}(i\to j)}}{\prod_{j:j\to i} x_j^{\#\,\mathrm{arr}(j\to i)}}.$$

Let $\mathcal{A}$ and $\overline{\mathcal{A}}$ be two cluster algebras, both of the same rank r, and with respective groups $\mathbb{P}$ and $\overline{\mathbb{P}}$ of Laurent monomials in the frozen variables. Then an algebra homomorphism $f : \mathcal{A} \to \overline{\mathcal{A}}$ that satisfies $f(\mathbb{P}) \subseteq \overline{\mathbb{P}}$ is called a *quasi-homomorphism* from $\mathcal{A}$ to $\overline{\mathcal{A}}$ if there are seeds $\Sigma = ((x_1, \ldots, x_s), Q)$ and $\overline{\Sigma} = ((\bar{x}_1, \ldots, \bar{x}_{\bar{s}}), \bar{Q})$ for $\mathcal{A}$ and $\overline{\mathcal{A}}$, such that

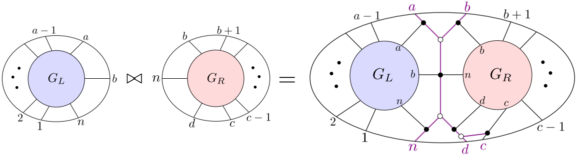
1. $f(x_i) \propto \bar{x}_i$ for $1 \le i \le r$
2. $f(\hat{y}_\Sigma(x_i)) = \hat{y}_{\overline{\Sigma}}(\bar{x}_i)$ for $1 \le i \le r$.
3. the map $i \mapsto \bar{i}$ of mutable nodes in Q and $\bar{Q}$ extends to an isomorphism of the corresponding induced subquivers.

Cluster adjacency theorem for BCFW tiles (EZ-L-P-SB-T-W)

Let $Z_{\mathfrak{r}}$ be a general BCFW tile of $\mathcal{A}_{n,k,4}(Z)$. Then for each facet $Z_{\mathfrak{r}'}$ of $Z_{\mathfrak{r}}$, there is a functionary $F_{\mathfrak{r}'}(\langle\langle I \rangle\rangle)$ which vanishes on $Z_{\mathfrak{r}'}$, such that the set

$$\{F_{\mathfrak{r}'}(\langle I \rangle) : Z_{\mathfrak{r}'} \text{ a facet of } Z_{\mathfrak{r}}\}$$

is a collection of compatible cluster variables for $\mathrm{Gr}_{4,n}$.



Proof idea:

- Have recursive construction of BCFW tiles by BCFW product.
- Each "facet functionary" for tile of $G_L \bowtie G_R$ is either image of facet functionary of G_L or G_R under product promotion, or is $\langle\langle I \rangle\rangle$ for $I \in \binom{\{a,b,c,d,n\}}{4}$.
- Product promotion is a cluster quasi-homomorphism, so cluster vars/clusters go to cluster vars/clusters (up to frozens).

Further cluster algebra connections for the amplituhedron

We associate to each general BCFW tile $Z_{\mathfrak{r}}$ a larger collection $x(\mathfrak{r})$ of compatible cluster variables of $Gr_{4,n}$ (including the "facet functionaries").

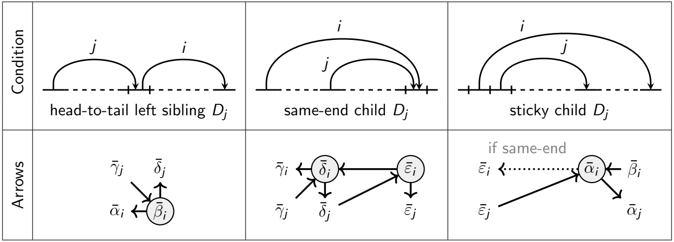
Sign description of BCFW tiles (EZ-L-P-SB-T-W)

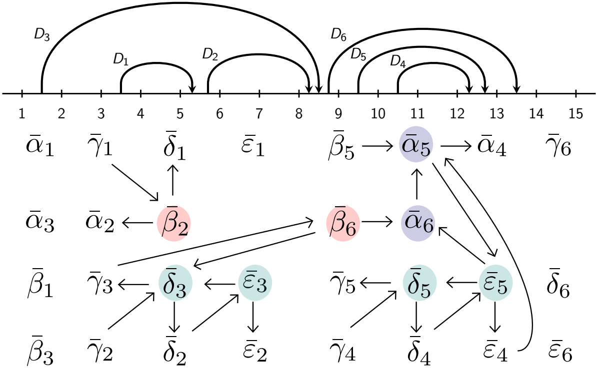
Let $Z_{\mathfrak{r}}$ be a general BCFW tile. For each element x of $x(\mathfrak{r})$, the functionary $x(Y)$ has a definite sign s_x on $Z_{\mathfrak{r}}^{\circ}$ and

$$Z_{\mathfrak{r}}^{\circ} = \{Y \in \mathrm{Gr}_{k,k+4} : s_x\, x(Y) > 0 \text{ for all } x \in x(\mathfrak{r})\}.$$

Analogous to fact that the totally positive Grassmannian is the subset of the Grassmannian where certain cluster variables are positive.

Furthermore, in the case of a standard BCFW tile (i.e. coming from a chord diagram), we have an explicit "local" description of all these compatible cluster variables as well as their quiver.


Condition
Arrows
j
i
head-to-tail left sibling D_j
same-end child D_j
sticky child D_j
if same-end
$\bar{\gamma}_j$ $\bar{\delta}_j$ $\bar{\alpha}_i$ $\bar{\beta}_i$
$\bar{\gamma}_i$ $\bar{\delta}_i$ $\bar{\varepsilon}_i$ $\bar{\gamma}_j$ $\bar{\delta}_j$ $\bar{\varepsilon}_j$
$\bar{\varepsilon}_i$ $\bar{\alpha}_i$ $\bar{\beta}_i$ $\bar{\varepsilon}_j$ $\bar{\alpha}_j$


D_3 D_1 D_2 D_6 D_5 D_4
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
$\bar{\alpha}_1$ $\bar{\gamma}_1$ $\bar{\delta}_1$ $\bar{\varepsilon}_1$ $\bar{\beta}_5$ $\bar{\alpha}_5$ $\bar{\alpha}_4$ $\bar{\gamma}_6$
$\bar{\alpha}_3$ $\bar{\alpha}_2$ $\bar{\beta}_2$ $\bar{\beta}_6$ $\bar{\alpha}_6$
$\bar{\beta}_1$ $\bar{\gamma}_3$ $\bar{\delta}_3$ $\bar{\varepsilon}_3$ $\bar{\gamma}_5$ $\bar{\delta}_5$ $\bar{\varepsilon}_5$ $\bar{\delta}_6$
$\bar{\beta}_3$ $\bar{\gamma}_2$ $\bar{\delta}_2$ $\bar{\varepsilon}_2$ $\bar{\gamma}_4$ $\bar{\delta}_4$ $\bar{\varepsilon}_4$ $\bar{\varepsilon}_6$

Explicit description of cluster variables

Theorem (Even-Zohar–Lakrec–Parisi–Sherman-Bennett–Tessler–W)

Have an explicit description of cluster variables (as "chain polynomials") and cluster containing each BCFW tile of $\mathcal{A}_{n,k,4}(Z)$.

Notation for quadratic function in Plücker coordinates:

$$\langle a\, b\, c \,|\, d\, e \,|\, f\, g\, h \rangle \;=\; \langle a\, b\, c\, d \rangle \langle e\, f\, g\, h \rangle - \langle a\, b\, c\, e \rangle \langle d\, f\, g\, h \rangle \qquad (1)$$

More generally, define the *chain polynomials* of degree $k+1$ by:

$$\langle a_0\, b_0\, c_0 \,|\, x_1^0\, x_1^1 \,|\, b_1\, c_1 \,|\, x_2^0\, x_2^1 \,|\, b_2\, c_2 \,|\, \ldots \,|\, x_k^0\, x_k^1 \,|\, b_k\, c_k\, d_k \rangle$$

$$= \sum_{t \in \{0,1\}^k} (-1)^{\sum t_i} \langle a_0\, b_0\, c_0\, x_1^{t_1} \rangle \langle x_1^{1-t_1}\, b_1\, c_1\, x_2^{t_2} \rangle \langle x_2^{1-t_2}\, b_2\, c_2\, x_3^{t_3} \rangle \cdots \langle x_k^{1-t_k}\, b_k\, c_k\, d_k \rangle$$

Cluster variables from chord diagram: roughly five cluster variables get associated to each chord c in chord diagram, and formula for each cluster variable associated to c (a chain polynomial) depends on chords above c.

Explicit description of cluster variables

Theorem 8.3 (Domino cluster variables as chain polynomials). *Let $D \in \mathcal{CD}_{n,k}$ be a chord diagram with chords $(a_1, b_1, c_1, d_1), \ldots, (a_k, b_k, c_k, d_k)$, and consider any chord $D_i = (a_i, b_i, c_i, d_i)$. The domino cluster variables $\mathbf{x}(D)$ are given by the following chain functionaries:*

$$\bar{\alpha}_i = \langle b_i\, c_i\, d_i \nearrow_i n\rangle$$

$$\bar{\beta}_i = \begin{cases} \langle a_i\, c_i\, d_i \nearrow_i n\rangle & \text{if } D_i \text{ not sticky} \\ \langle a_i'\, a_i\, c_i\, d_i\rangle & \text{if } D_i \text{ is a sticky but not same-end child} \\ \bar{\alpha}_p & \text{if } D_i \text{ is a sticky same-end child of another chord } D_p \end{cases}$$

$$\bar{\gamma}_i = \begin{cases} \langle n \nwarrow_i a_i\, b_i \,|\, c_i\, d_i \,|\, c_j\, d_j \nearrow_i n\rangle & \text{if } D_i \text{ has sibling } D_j = (c_i, d_i, c_j, d_j) \text{ and } D_i \text{ not sticky} \\ \langle a_i\, a_i'\, b_i \,|\, c_i\, d_i \,|\, c_j\, d_j \nearrow_i n\rangle & \text{if } D_i \text{ has sibling } D_j = (c_i, d_i, c_j, d_j) \text{ and } D_i \text{ sticky} \\ \langle n \nwarrow_p a_p\, b_p \,|\, a_i\, b_i \,|\, c_i\, d_i \nearrow_p n\rangle & \text{if } D_i \text{ has same-end parent } D_p\text{, and } D_i, D_p \text{ not sticky} \\ \langle a_p\, a_p'\, b_p \,|\, a_i\, b_i \,|\, c_i\, d_i \nearrow_p n\rangle & \text{if } D_i \text{ has same-end parent } D_p,\ D_i \text{ not sticky, } D_p \text{ sticky} \\ \langle a_i'\, a_i\, b_i \nearrow_p n\rangle & \text{if } D_i \text{ has same-end parent } D_p,\ D_i \text{ sticky, } D_p \text{ not sticky} \\ \langle a_p'\, a_i'\, a_i\, b_i\rangle & \text{if } D_i \text{ has same-end parent } D_p\text{, and } D_i, D_p \text{ sticky} \\ \langle a_i\, b_i\, d_i \nearrow_i n\rangle & \text{otherwise, if } D_i \text{ not sticky} \\ \langle a_i'\, a_i\, b_i\, d_i\rangle & \text{otherwise, if } D_i \text{ sticky} \end{cases}$$

$$\bar{\delta}_i = \begin{cases} \langle a_i\, b_i\, c_i \nearrow_i n\rangle & \text{if } D_i \text{ not sticky} \\ \langle a_i'\, a_i\, b_i\, c_i\rangle & \text{if } D_i \text{ sticky} \end{cases}$$

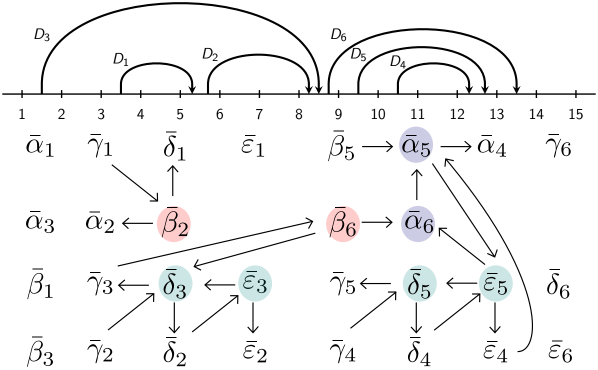
$$\bar{\varepsilon}_i = \langle a_i\, b_i\, c_i\, d_i\rangle$$

where we use the following notation for pieces of chain functionaries:

$$|\ldots x\, y \nearrow_i n\rangle = |\ldots x\, y \,|\, b_{(1)}\, a_{(1)} \,|\, c_{(1)}\, d_{(1)} \,|\, b_{(2)}\, a_{(2)} \,|\, c_{(2)}\, d_{(2)} \,|\, \cdots \,|\, b_{(m)}\, a_{(m)} \,|\, c_{(m)}\, d_{(m)}\, n\rangle$$
$$\langle n \nwarrow_i x\, y \ldots| = \langle n\, c_{(m)}\, d_{(m)} \,|\, b_{(m)}\, a_{(m)} \,|\, \cdots \,|\, c_{(2)}\, d_{(2)} \,|\, b_{(2)}\, a_{(2)} \,|\, c_{(1)}\, d_{(1)} \,|\, b_{(1)}\, a_{(1)} \,|\, x\, y \ldots|,$$

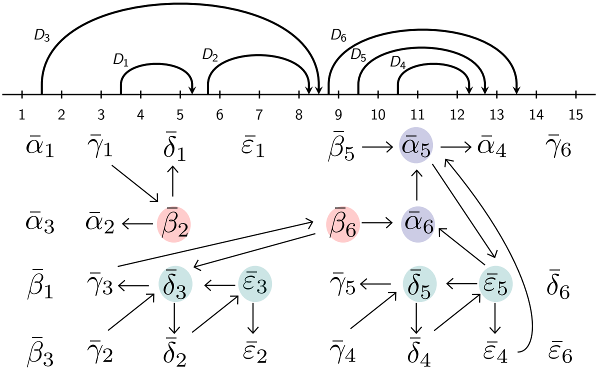
where the chords $D_{(1)} = (a_{(1)}, b_{(1)}, c_{(1)}, d_{(1)}), \ldots, D_{(m)} = (a_{(m)}, b_{(m)}, c_{(m)}, d_{(m)})$ are the following possibly-empty chain of ancestors of D_i, ordered bottom to top: $D_{(1)}$ is the lowest ancestor of D_i that does not end in (x, y), and $D_{(r+1)}$ is the lowest ancestor of $D_{(r)}$ that does not end at $(c_{(r)}, d_{(r)})$, i.e. is not same-end with $D_{(r)}$.

Example



D_3
D_1
D_2
D_6
D_5
D_4
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
$\bar\alpha_1$ $\bar\gamma_1$ $\bar\delta_1$ $\bar\varepsilon_1$ $\bar\beta_5$ $\bar\alpha_5$ $\bar\alpha_4$ $\bar\gamma_6$
$\bar\alpha_3$ $\bar\alpha_2$ $\bar\beta_2$ $\bar\beta_6$ $\bar\alpha_6$
$\bar\beta_1$ $\bar\gamma_3$ $\bar\delta_3$ $\bar\varepsilon_3$ $\bar\gamma_5$ $\bar\delta_5$ $\bar\varepsilon_5$ $\bar\delta_6$
$\bar\beta_3$ $\bar\gamma_2$ $\bar\delta_2$ $\bar\varepsilon_2$ $\bar\gamma_4$ $\bar\delta_4$ $\bar\varepsilon_4$ $\bar\varepsilon_6$

Thank you for listening!



- "Cluster algebras and the $m = 4$ amplituhedron,"
  Even-Zohar, Lakrec, Parisi, Sherman-Bennett, Tessler, Williams arXiv:2310.17727