18.336 04/23/2013 Multigrid. (geometric)

	Apply int. op.	Solve PDE	Solve Int. eq.
Uniform	FFT	FFT/DST	Krylov, FFT
Nonuniform	Precorr. FFT	Multigrid	
		H-matrices	
Very nonuniform	FMM		
HF nonunif.	butterfly	sweeping	?

Multigrid for $Lu = f$, $x \in \Omega$

and/or $L = L(x)$

$\Leftrightarrow u = Gf$ but G unavailable

ex: $-\nabla \cdot \alpha(x) \nabla u = f$

	M	FMM
R	restriction (f)	projection (f)
P	prolongation/interpolation (u)	interpolation (u)

Idealized algo

$u = \text{Solve}_N(L, f)$

- restrict L, f to coarse scale $\to \tilde{L}, \tilde{f}$
- $\tilde{u} = \text{Solve}_{N/2}(\tilde{L}, \tilde{f})$
- interpolate $\tilde{u}$ to fine scale $\to u$

Assume $[v, r] = \text{ApproxSolve}_N(L, f; u_0)$ so $\begin{cases} Lv = f + r \\ Lu = f \end{cases}$

initial

$L\underbrace{(v - u)}_{-e} = r$

$\left(\begin{array}{l} \text{if } e = \text{Solve}(L, -r) \\ \text{then } u = v + e \end{array} \right)$

$u = \text{Solve}_N(L, f)$

- $[v, r] = \text{ApproxSolve}_N(L, f; 0)$
- restrict L, r to coarse scale $\to \tilde{L}, \tilde{r}$
- $\tilde{e} = \text{Solve}_{N/2}(\tilde{L}, -\tilde{r})$
- interpolate $\tilde{e}$ to fine scale $\to e^{\#}$ $(\neq e)$
- $u^{\#} = v + e^{\#}$
- $u^{*} = \text{ApproxSolve}_N(L, f; u^{\#})$

Called multigrid V-cycle.

$\to$ restriction of residuals, interpolation of errors

Q. | How to choose Approx Solve, R, I so that $u^{\#}$ is sufficiently close to u, or $\tilde{L}\tilde{e} = -\tilde{r}$ is informative about $Le = -r$.

ex. $\begin{cases} -u'' = f & , x \in [0,1] \\ u(0) = u(1) = 0. \end{cases}$

$$L = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{(N-1)\times(N-1)} \qquad L > 0$$

$[v, r] = \text{ApproxSolve}_N(L, f; u_0)$

Richardson (also called weighted Jacobi for this problem)

$$u_{k+1} = u_k + \omega(f - Lu_k)$$

$\hookrightarrow \omega > 0$

then $v = u_5$

Convergence prop: $e = u - u_k$

$$u - u_{k+1} = u - u_k - \omega(Lu - Lu_k)$$

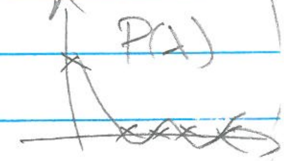
$$e_{k+1} = (I - \omega L)e_k$$

$$\Rightarrow e_k = (I - \omega L)^k e_0$$

(ex. Krylov: $e_{k+1} = P(L)e_0$

$\downarrow$ deg k

"best" with $\begin{cases} P(0) = I \\ \deg P = k \end{cases}$)



Need $\|I - \omega L\| < 1$

$$\lambda_m(I - \omega L) = 1 - \omega\lambda_m(L)$$

with $0 < \lambda_{min} \leq \lambda_m \leq \lambda_{max}$

$\lambda_{min} = \pi^2 + O(h^2)$, $\lambda_{max} \to = \frac{4}{h^2} + O(h^2)$

$\rightarrow$ need $-1 < 1 - \omega\lambda_m(L) < 1$

✓

$$\omega \lambda_m(L) < 2 \quad \forall j$$

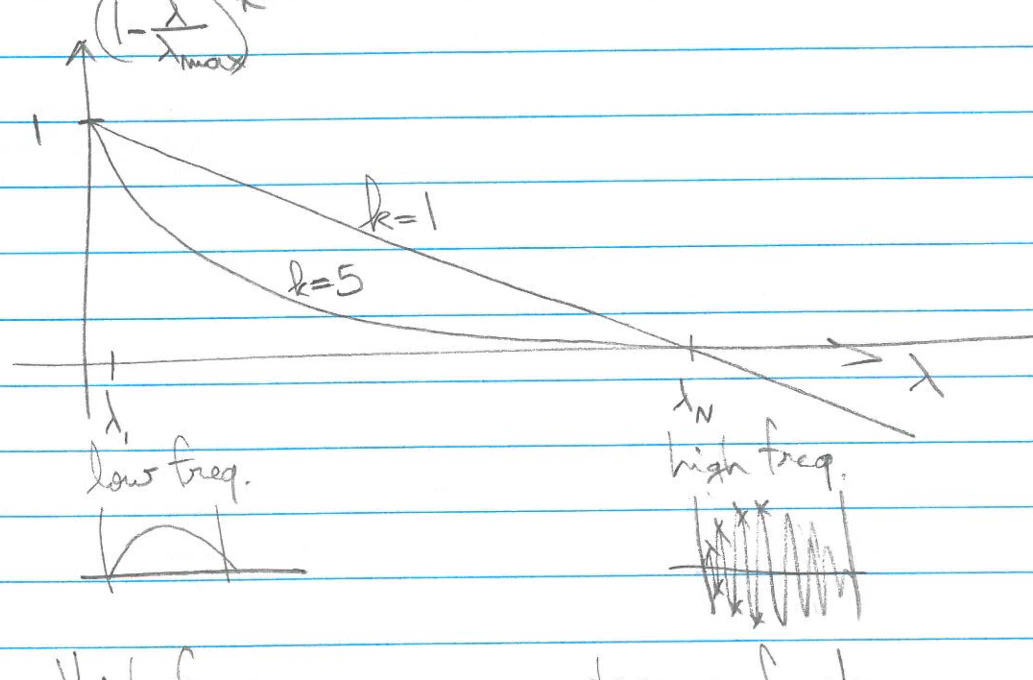
$$\omega < \frac{2}{\lambda_{max}(L)}, \quad \left(\text{presumably } \omega = \frac{1}{\lambda_{max}} \text{ is good}\right)$$

$$\begin{cases} L v_m = \lambda_m v_m & \text{with } v_m(j) = \sin(\pi m x_j) \\ e_k = (I - \omega L)^k e_0 \end{cases}$$

then $v_m^T e_k = v_m^T (I - \omega L)^k e_0$

$$= \underbrace{(1 - \omega \lambda_m)^k}_{\text{attenuation factor for the error in eigenspace } m.} v_m^T e_0$$

e.g. $\omega = \frac{1}{\lambda_{max}}$



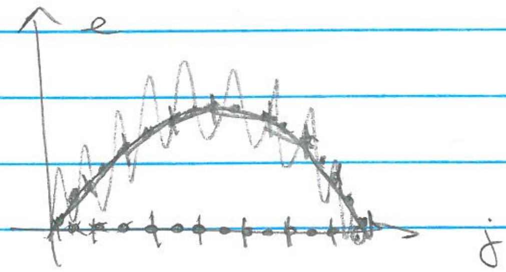
$\rightarrow \begin{cases} \text{high frequency error decays fast} \\ \text{low} \quad \text{———} \quad \text{slowly} \end{cases}$

$\rightarrow$ Weighted Jacobi, $\omega = \frac{1}{\lambda_{max}}$, is a <u>smoother</u>.

(Krylov is not!)

Also the case
- in higher dim
- provided $\alpha(x)$ not too rough
- other discretizations.

Residual is also smoothed: $Le = -r$



Idea: R = moving average & downsampling
I = PL interpolation

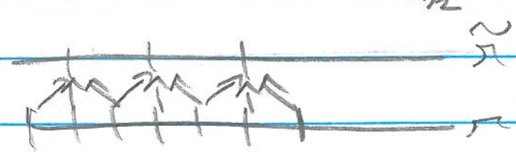
relative to coarse grid, $\sin(\pi m x)$ has twice the frequency it had on the fine grid:

repopulate high frequencies

$$\sin(\pi m j h) = \sin\left(\pi (2m) j \frac{h}{2}\right)$$

and $\tilde{L} = L_{N/2}$ (same discretized ODE)

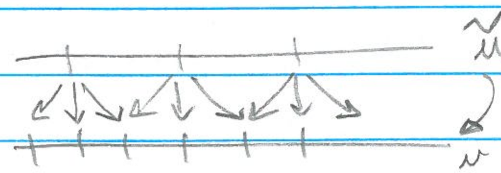
R:



$\tilde{r} = R r$

$$R = \frac{1}{2}\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & & & \\ & & \frac{1}{2} & 1 & \frac{1}{2} & \\ & & & & & \ddots \end{bmatrix}$$

I:



$u = I \tilde{u}$

$$I = \begin{bmatrix} 1 & & \\ 1/2 & 1/2 & \\ & 1 & \\ & 1/2 & 1/2 \\ & & \ddots \end{bmatrix} = 2R^T$$

For smooth functions, $\boxed{IR \simeq id}$

→ Multigrid relies on smoothing property of ApproxSolve ex in 2D : mask for R: $\frac{1}{2}[1\ 2\ 1]$

$$\to \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$ (moving average)

$I = 4R^T$: piecewise bilinear interpolation

$I = I_x \otimes I_y$

$R = R_x \otimes R_y$

Next:
- AMG / other geometries
- error analysis
- other cycles

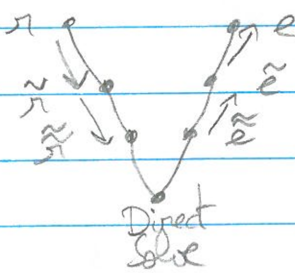
04/25/2013 : Convergence analysis $Au=f$, $A=A^*$, $A\succ 0$.
Given R, $I = cR^T$

Algo (V-cycle) $u = \text{V-cycle}(A, f; u_0)$

$u = \text{ApproxSolve}(A, f; u_0)$, get $r = f - Au$

Restrict $A, r \to \tilde{r} = Rr$, $\tilde{A} = RAI$ $\boxed{R}$

$\tilde{e} = \text{V-cycle}(\tilde{A}, \tilde{r}; 0)$ or direct solve if small



Interpolate $e = I\tilde{e}$ $\boxed{I} \to$ will call identity id.

$u := u + e$

$u := \text{ApproxSolve}(A, f; u)$

$\hookrightarrow$ Smoother (weighted Jac., GS, red-black G-S)

Complexity: $C(n) = \alpha n + C(n/2) \Rightarrow C(n) \le 2\alpha n$

Def. $\tilde{A} = RAI$, check for 1D Laplacian

2-grid cycle: $\tilde{e} = -\tilde{A}^{-1}\tilde{r}$ (actual solve)

$\Rightarrow u = Mu_0 + g$ and $(u - A^{-1}f) = M(u_0 - A^{-1}f)$

2-grid operator M: $u := Su_0$ (smoother)

$r = f - ASu_0$, $\tilde{r} = R(f - ASu_0)$,

$\tilde{e} = \tilde{A}^{-1}R(f - ASu_0)$, $u := Su_0 + I\tilde{A}^{-1}R(f - ASu_0)$

$$u = \underbrace{S(id - I\tilde{A}^{-1}RA)\tilde{S}}_{M} u_0 + \underbrace{SI\tilde{A}^{-1}Rf}_{g}$$

$$u - A^{-1}f = Mu_0 + SI\tilde{A}^{-1}RAA^{-1}f - A^{-1}f$$

$$= Mu_0 + S(I\tilde{A}^{-1}RA - id)\tilde{S}A^{-1}f$$

$\tilde{S}$ → smoother does not change the solution.

$$= M(u_0 - A^{-1}f)$$

Let $T = id - I\tilde{A}^{-1}RA$. (coarse-grid correction).

Show $\|ST\underbrace{SS}_{\tilde{S}}T\underbrace{SS}_{\tilde{S}}TS\cdots\| < 1$

Show $\|ST\| < 1$

smoother / coarse grid correction.

<u>Notations</u>. $\langle x, y\rangle = x^T y$

for any $A = A^*$, $A > 0$: $\langle x, y\rangle_A = \langle Ax, y\rangle$, let $(A^{1/2})^2 = A$.

$\|x\|_A = \sqrt{\langle Ax, x\rangle} = \|A^{1/2}x\|$ (large wrt $\|x\|$ when x is oscillatory)

$\|ST\|_A = \max_{x \neq 0} \|STx\|_A / \|x\|_A$.

<u>Lemma</u> (i) T is a projector: $T^2 = T$

Orthog: $Ker(T) = Ran(T)^{\perp}$

(ii) orthogonal w.r.t. A: $\langle Tx, y\rangle_A = \langle x, Ty\rangle_A$

(iii) $Ran(T)$ orthog. $Ran(I)$

<u>Pf.</u> (i) STP $id - T = I\tilde{A}^{-1}RA$ is a projector:

$$I\tilde{A}^{-1}\underbrace{RAI}_{\tilde{A}}\tilde{A}^{-1}RA = I\tilde{A}^{-1}RA = id - T$$ ✓

(ii) $\langle A(id-T)x, y\rangle = \langle AI\tilde{A}^{-1}RAx, y\rangle = \langle Ax, I\tilde{A}^{-1}RAy\rangle = \langle Ax, (id-T)y\rangle$

(iii) Show $\langle ATx, Iy\rangle = 0 \ \forall x, y$

$\Leftrightarrow \langle RATx, y\rangle = 0 \ \forall x, y$

$\Leftrightarrow RAT = 0$

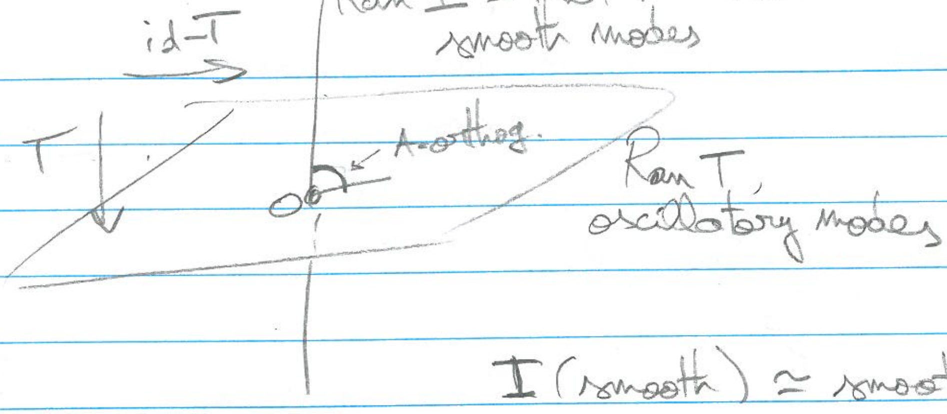
$RA(id - I\tilde{A}^{-1}RA) = RA - \tilde{A}\tilde{A}^{-1}RA = 0$. □

③

$$\operatorname{Ran} I = \operatorname{Ker} T^{*} = \operatorname{Ker} T = \operatorname{Ran}(id - T)$$

A-adjoint

$T = T^{*}$



$I(\text{smooth}) \simeq \text{smooth}$

$T(\text{smooth}) \simeq 0 \quad \Rightarrow$ coarse-grid corr. works to eliminate the smooth error.

<u>Lemma</u> (Smoothing property of S) For some $\alpha > 0$,

$$\|Se\|_A^2 \le \|e\|_A^2 - \alpha \|e\|_{A^2}^2 \qquad \forall e$$

<u>Pf</u> For $S = id - \omega A$ (Richardson), $\omega = \frac{1}{\lambda_{max}(A)}$

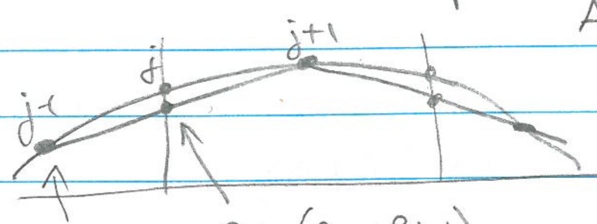
$$\begin{aligned}
\|Se\|_A^2 &= \langle A(id - \omega A)e, (id-\omega A)e\rangle \\
&= \langle Ae, e\rangle - 2\omega\langle A^2 e, e\rangle + \omega^2 \langle A^2 e, Ae\rangle \\
&= \|e\|_A^2 - \langle \omega(2id - \omega A)Ae, Ae\rangle \\
&\le \|e\|_A^2 - \underbrace{\lambda_{min}(\omega(2id - \omega A))}_{2\omega - \omega^2\lambda_{max}(A) = \omega = \alpha} \|e\|_{A^2}^2 \qquad \square
\end{aligned}$$

<u>Def</u>. (Approximation property of I)

$$\min_{\tilde{e}} \|e - I\tilde{e}\|^2 \le \beta \|e\|_A^2 \qquad \text{for some } \beta > 0$$

$\hookrightarrow \|Ae\|^2$

<u>Pf</u>. (I = PL interp., $\tilde{e}$ = coarse downsampling)
$A = -d^2/dx^2$, centered



$$e_j - \left(\frac{e_{j+1} + e_{j-1}}{2}\right) = 2h^2\left(\frac{-e_{j-1} + 2e_j - e_{j+1}}{h^2}\right) \Rightarrow \beta = \sqrt{2}\, h.$$

Thm. Assume (Smoothing prop) & (Approx prop.) hold for some $\alpha > 0$, $\beta > 0$. Then

(i) $\alpha \le \beta$

(ii) two-level iteration converges with

$$\|ST\|_A \le \sqrt{1 - \frac{\alpha}{\beta}}$$

Pf. Pick $e \in \mathrm{Ran}(T)$ on fine grid (any)

$\tilde{e}$ on coarse grid (TBD)

Then $\langle e, I\tilde{e} \rangle_A = 0$ because $\mathrm{Ran}\,T \perp \mathrm{Ran}\,I$.

$e \in \mathrm{Ran}\,T$, $I\tilde{e} \in \mathrm{Ran}\,I$

$$\Rightarrow \|e\|_A^2 = \langle Ae, e \rangle = \langle Ae, e - I\tilde{e} \rangle$$

$$\le \|Ae\| \, \|e - I\tilde{e}\| \qquad \text{(Cauchy-Schwarz)}$$

$\min_{\tilde{e}} \Rightarrow$ $\|e\|_A^2 \le \|Ae\| \sqrt{\beta} \|e\|_A$

$$\|e\|_A \le \sqrt{\beta} \|Ae\| \qquad \forall e \in \mathrm{Ran}\,T$$

$$\|Te\|_A \le \sqrt{\beta} \|ATe\| \qquad \forall e.$$

Now $0 \le \|STe\|_A^2 \le \|Te\|_A^2 - \alpha \|ATe\|^2$

$$\le \|Te\|_A^2 - \frac{\alpha}{\beta} \|Te\|_A^2$$

$$= \left(1 - \frac{\alpha}{\beta}\right) \|Te\|_A^2$$

$$\le \left(1 - \frac{\alpha}{\beta}\right) \|e\|_A^2$$

projector, because $\|e\|_A^2 = \|Te\|_A^2 + \|(id - T)e\|_A^2$

$\Rightarrow 1 - \frac{\alpha}{\beta} \ge 0 \Rightarrow \alpha \le \beta$

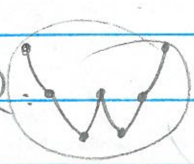
and $\|STe\|_A \le \sqrt{1 - \frac{\alpha}{\beta}} \|e\|_A \quad \forall e$ $\square$

Next. W-cycle:

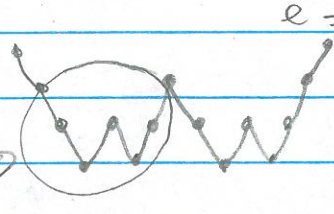
$\tilde{e} = \text{V-cycle}(\tilde{A}, \tilde{r}; 0) \rightarrow \tilde{e}_1 = \text{W-cycle}(\tilde{A}, \tilde{r}; 0)$

$\tilde{e} = \text{W-cycle}(\tilde{A}, \tilde{r}; \tilde{e}_1)$

2-level:



3-level:



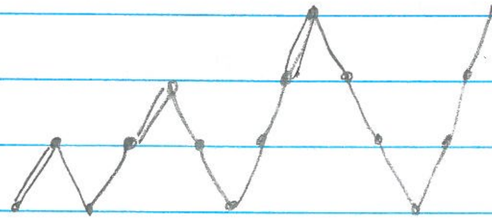
backside

For $l=1,\dots,p$ (so that $2^{-p}h_0$ is finest)

$u := I u$ (go to finer scale)

$u := MG(A, u, f^h)$

end

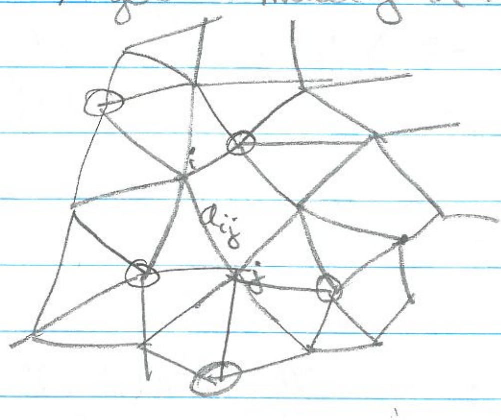


FMG, V-cycle

FMG, W-cycle

Analysis of FMG from $\|ST\|_A < 1$:
Iterative methods for sparse linear systems,
Y. Saad, p. 430.

Algebraic multigrid: nested graphs,



$$(Ae, e) = \sum_{i\sim j} a_{ij} e_i e_j$$

Smoothness of e: $(Ae, e) \ll \|e\|^2$

$\Rightarrow Ae \simeq 0 \Rightarrow a_{ii} e_i \simeq -\sum_{\substack{j\sim i\\ j\neq i}} a_{ij} e_j$ ← either connected to, or equal to

$j \in N_i = C_i \cup F_i$ (coarse, fine)

$\Rightarrow a_{ii} e_i \simeq -\sum_{j\in C_i} a_{ij} e_j - \sum_{j\in F_i} a_{ij} e_j$

Interpolation: $\left(a_{ii} + \sum_{j\in F_i} a_{ij}\right) e_i = -\sum_{j\in C_i} a_{ij} e_j$ (simplest)