

18.336J/6.335J :: Homework 1 :: Spring 2012 :: Due February 23

NOTE: As this is the first time 18.336 is focussing on fast solvers, the problem sets are new and may contain unexpectedly difficult to interpret questions or have unanticipated solutions and problems. Please do not hesitate to ask for clarifications by e-mail or at office hours, laurent@math.mit.edu or white@mit.edu.

1. (5 pts) Consider the boundary-value problem

$$-u''(x) - a_\epsilon(x)u(x) = f(x), \quad x \in [0, 1], \quad u(0) = u(1),$$

where $a(x)$ is some perturbation of the constant 1. For the numerical tests below, consider both

$$a_{\epsilon,1}(x) = 1 + \epsilon \cos(2\pi x), \quad \text{and} \quad a_{\epsilon,2}(x) = 1 + \epsilon x.$$

Let L_ϵ be the simple 3-point FD discretization of $-\frac{d^2}{dx^2} - a_\epsilon(x)$ with periodic boundary conditions.

- (a) Check numerically that the discrete Fourier transform F diagonalizes L_0 in the case $\epsilon = 0$, namely, $F^*L_0F = \Lambda$ where Λ is diagonal. Plot the behavior of $\min(\Lambda)$ and $\max(\Lambda)$ as a function of $N = \frac{1}{h}$.
- (b) As a function of $\epsilon \neq 0$, show numerically how far $F^*L_\epsilon F$ is from a diagonal matrix. Show numerically how far F is from the matrix P of eigenvectors of L_ϵ .
- (c) (Challenging) Is the “preconditioned” system, $L_0^{-1}L_{0.1}u = L_0^{-1}f$ easy to solve?
2. (2 pts) Consider a matrix A with eigenvalues $\lambda_{A,j}$, and a matrix B with eigenvalues $\lambda_{B,j}$. Using properties of the Kronecker product, show that $I \otimes A + B \otimes I$ has eigenvalues $\lambda_{A,j} + \lambda_{B,k}$.
3. (6 pts) For each of the following problems, (i) implement and validate a fast algorithm, and (ii) show a representative computer plot of the corresponding eigenvalues and eigenvectors.

(a)

$$u'''' = f, \quad x \in [0, 1], \quad u(0) = u(1), \quad u'(0) = u'(1), \quad u''(0) = u''(1).$$

(b)

$$-u'' + 10u' = f, \quad x \in [0, 1], \quad u(0) = u(1).$$

(c)

$$-u'' = f, \quad x \in [0, 1], \quad u'(0) = 0, \quad u(1) = 0.$$

4. (7 pts) Consider the basic second-order FD discretization for the Poisson problem on the square $[0, 1]^2$ with homogeneous Dirichlet boundary conditions,

$$-\Delta u(x, y) = f(x, y), \quad u(x, y) = 0 \text{ when either } x = 0, 1 \text{ or } y = 0, 1.$$

- (a) Implement a solver for the discretized system using Matlab’s sparse direct solver (see Matlab code on the class website). When using the sparse direct solver, which matlab command is faster $u = L \setminus b$ or $u = \text{inv}(L) * b$? and why?
- (b) Implement another solver for the same system using the discrete sine transform DST-I. Choose some nonzero right-hand side and some grid spacing h . Give evidence that your direct solver and DST-I solvers produce answers that match to near machine precision.
- (c) Plot the runtimes of both methods in log-log scale as a function of $N = 1/h$.
- (d) The asymptotic complexity, as $N \rightarrow \infty$, of matlab’s algorithm for sparse Gaussian elimination is known analytically (at least for the case of 2-D grids and no numerical pivoting). Compare the analytic result (Google is your friend) to the scalings you obtained in point (c).

You will find code for the DST-I and the implementation of the 2D Laplacian as a sparse matrix on the class webpage.