

18.336 projects :: Spring 2010

*: limited-scope class project

***: can lead to interesting research, but does not have to.

***: research project

1. **Meshfree finite differences (**)**. It is sometimes possible to very accurately estimate derivatives of a function from point samples $f(x_j)$, even if the x_j are not on a grid. This involves solving small linear systems for the weights w_{ij} in a formula such as $f'(x_i) \simeq \sum_j w_{ij} f(x_j)$. In this project you will implement such finite difference schemes numerically for the Laplacian on 2D random point clouds, give an account of the convergence theory, and test convergence as the cloud gets denser. As an application, you will solve the heat equation. (***) Additionally, you may test the extent to which arranging the points near certain algebraic curves affects the conditioning of the systems involved, and the probability with which this happens in random point clouds.
2. **Nonnormal dynamics (*)**. The stability analysis of finite difference schemes for time-dependent problems is sometimes complicated by the presence of boundary conditions. In this project you will implement the 1D advection-diffusion equation with Dirichlet B.C. using an upwind scheme, present and illustrate the stability analysis for this scheme, and interpret your findings physically in terms of nonnormal matrices and transient growth (that you will quantify.) Finally, you will briefly summarize the findings of a survey paper that link such “nonnormal effects” to convective instabilities in fluid dynamics.
3. **Perfectly matched layer (**)**. In this project you will implement and test the method called “perfectly matched layer” for realizing an absorbing boundary layer for the wave equation. You will quantify the relationships between grid spacing, time step, layer width, and layer strength which make the method provably effective. (***) In addition to the above, you may document the difficulties associated with (1) the presence of right-hand side oscillatory forcings, and (2) media that are inhomogeneous in a way that the parameter gradients are not parallel to the boundary.
4. **Chebyshev symbols (***)**. Spectral methods efficiently represent smooth functions as Chebyshev polynomial expansions, but it turns out that similar ideas potentially apply to operators as well. In this project, you will study the so-called Chebyshev-symbolic representation of both $\mathcal{L} = \nabla \cdot \alpha(x) \nabla$ and its inverse in a square, with Dirichlet B.C., and smooth $\alpha(x)$. This representation takes a form akin to

$$\mathcal{L}^{-1}f(x) = \sum_{n \geq 0} \begin{pmatrix} T_n(x) & U_n(x) \end{pmatrix} \sigma(x, n) \begin{pmatrix} \langle f, T_n \rangle \\ \langle f, U_n \rangle \end{pmatrix},$$

where for each x and n , $\sigma(x, n)$ is a 2-by-2 matrix. You make this idea precise. You will quantify, numerically and perhaps also analytically, the separation rank of the components of σ in x vs. n . This new representation is an extension of “discrete symbol calculus” which will have applications for operator-based fast algorithms for PDE in variable media.

5. **Waves in discontinuous media (**)**. Finite differences methods cannot solve the wave equation very accurately in a discontinuous medium when the grid is Cartesian. In this project you will document these difficulties in a quantitative manner for the leap-frog scheme for the 1D wave equation, both in terms of order of convergence, and in terms of shifts in wavefront arrivals. (***) Additionally, you may try to fix these problems by performing an adaptive change of variables to better resolve the singularities, and passing to an implicit timestepping scheme.
6. **The fast marching method (*)**. The distance function $\phi(x)$ to any (nice) set Ω obeys the eikonal equation $\|\nabla\phi(x)\| = 1$, with boundary condition $\phi(x) = 0$ when $x \in \partial\Omega$. For this project you will implement the so-called fast marching method for solving the eikonal equation, you will present the accuracy analysis of your scheme, and you will discuss the difficulties that arise when $\nabla\phi$ is discontinuous.

7. **The level set method** (*). One problem in computational fluid dynamics is to track the interface between two immiscible liquids, like oil and water. One efficient way of encoding this interface numerically is as the zero level set of a function that came to be called a “level-set” function. In this project, you will use the level set method to simulate an immersed bubble that deforms under the action of gravity. Mass is not conserved numerically inside the bubble: you will document this mass loss problem.
8. **Application-oriented project**. Alternatively, you may choose a PDE from some application, implement an adequate numerical method for it, and discuss its convergence as the mesh size tends to zero. Recycling an existing project is not allowed. Requires the consent of the instructor. Here are some ideas:
- Black-Scholes model (finance)
 - Elastic wave propagation (geophysics, civil eng.)
 - Two-phase flows in porous media (geophysics)
 - Hamilton-Jacobi-Bellman equation (stochastic control)
 - Stellar dynamics (physics, relativity)
 - Schroedinger equation (physics, quantum mechanics)
 - Reactive flows (mechanical eng.)
 - Eddy viscosity models for turbulence (mechanical eng.)
 - Gas dynamics and shocks (aeronautics)
 - Atmospheric flows (enviro. eng.)
 - Oceanic flows, Coriolis (enviro. eng.)
 - Glaciers (enviro. eng.)
 - Neutron transport (nuclear eng.)
 - etc.