

18.336 Homework 2 :: Spring 2011 :: Due March 17

1. (20 pts) Consider the function

$$f(x) = e^{-400(x-0.5)^2}$$

for $x \in [0, 1]$. Sample it on a grid $x_j = jh$ with $h = 1/N$ and $0 \leq j \leq N$, for some N to be determined. Consider the linear interpolant of f computed from the $N + 1$ samples $f(x_j)$. Numerically, find the smallest value of N such that f differs from its linear interpolant by at most 10^{-2} in the uniform norm. [Hint: Matlab has `interp1.m` for 1D linear interpolation.]

2. Consider the 2D wave equation

$$u_{tt} = \Delta u, \quad 0 \leq x, y \leq 1.$$

Use homogeneous Dirichlet boundary conditions. Fix the initial conditions to be

$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = f(x)f(y),$$

where f was defined in problem 1. Consider a spatial grid $\mathbf{x}_j = (x_{j_1}, y_{j_2}) = (j_1\Delta x, j_2\Delta x)$ with Δx small enough to resolve the initial condition, in the sense of problem 1. (i.e., take Δx less than $h = 1/N$, where the critical N was found in problem 1.)

- (a) (40 pts) Implement and test the “simplest” numerical method, which uses the 3-point formula for the second derivative in time, and the 5-point Laplacian at time t_n . It results in a two-step method. Explain how you initialize your scheme. Show a log-log plot of the error vs. the grid spacing Δx , and check from this plot that your method is second-order accurate.
- (b) (10 pts) Consider the ODE $y''(t) = \lambda y$, and the 3-point rule for y'' as a two-step explicit time integrator. Find the region of stability of this ODE solver in terms of $\lambda(\Delta t)^2$, and plot it in the complex plane.
- (c) (10 pts) From your answer to (b), and your knowledge of the spectrum of the discrete Laplacian, perform the “method of lines” stability analysis for the method in (a). What CFL condition does this analysis result in?
- (d) (10 pts) Perform the von Neumann stability analysis for the method in (a), and check if the resulting CFL condition agrees with what you found in the previous question. [Hint: since this is a 2D problem, a plane wave is $\exp(ik_1j_1\Delta x) \exp(ik_2j_2\Delta x)$.]
- (e) (10 pts) Find the modified equation that corresponds to the numerical method in (a). Solve it via Fourier series, and comment on the physics of the extra terms. Are they dissipative, dispersive, or something else?
3. (Bonus, 20 pts) Formulate and prove one extension of the Lax equivalence theorem to the case of linear ODE with two time derivatives.

Homework 3 will have a question that requires comparing the finite difference schemes developed here with a spectral method, so please consider saving your scripts.