18.336 Homework 2 hints :: Spring 2010

- Question 2 – setup. The boundary conditions for u and v may look complicated, but they are the only ones that are compatible with those for p. Disregard the fact that f(x)f(y) does not vanish exactly at the boundary – it does to good accuracy. You get partial credit for implementing periodic boundary conditions instead of those specified (however you will be slightly handicapped in homework 3 where it will be asked to benchmark your code.)

- Question 2(c). There should be one amplitude to keep track of per equation, so the plane wave solution should be written as
$$\begin{pmatrix} U_j^n \\ V_j^n \\ P_j^n \end{pmatrix} = \begin{pmatrix} \rho_U^n \\ \rho_V^n \\ \rho_P^n \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x_j}}.$$
Since there are two spatial dimensions, $\mathbf{k}$ means (k_1, k_2), and $\mathbf{k}\cdot\mathbf{x_j} = k_1 j_1 \Delta x + k_2 j_2 \Delta x$. The amplification "factor" is now a 3-by-3 matrix $G(\mathbf{k})$, such that
$$\begin{pmatrix} U_j^{n+1} \\ V_j^{n+1} \\ P_j^{n+1} \end{pmatrix} = G(\mathbf{k}) \begin{pmatrix} U_j^n \\ V_j^n \\ P_j^n \end{pmatrix}.$$
Stability means that the eigenvalues of $G(\mathbf{k})$ are less than or equal to one in modulus.

- Question 4. A two-step method is initialized from the solution at steps $n = 0$ and $n = 1$, but the initial condition only gives $n = 0$. In practice, this is resolved by computing the solution accurately to time $t^1 = \Delta t$ by some other method, in such a way that the one-step error committed is of the same order as that of leap-frog, $O((\Delta t)^3)$. This can be done by taking several tiny steps of a first-order method, or one step by a higher-order method like Runge-Kutta 2. The question of stability does not pose itself for this first step (why?).

- Question 4. Since the scheme relates the solution at steps $n-1$, n, and $n+1$, the von Neumann analysis is more complicated. You can either write a quadratic equation for the 3-by-3 amplification matrix, or you can consider that your vector of unknowns has 6 components corresponding to two consecutive times. For instance, you may write
$$\begin{pmatrix} U_j^{n+1} \\ V_j^{n+1} \\ P_j^{n+1} \\ U_j^n \\ V_j^n \\ P_j^n \end{pmatrix} = G(\mathbf{k}) \begin{pmatrix} U_j^n \\ V_j^n \\ P_j^n \\ U_j^{n-1} \\ V_j^{n-1} \\ P_j^{n-1} \end{pmatrix},$$
and use a similar plane wave solution as above, with 6 amplitudes. Note that $G(\mathbf{k})$ is 6-by-6, with a 2-by-2 block structure. The lower-left block should be the identity. Don't waste your time computing the determinant of the 6-by-6 matrix to calculate the eigenvalues; since the matrix is sparse you can start by eliminating the bottom three unknowns in $(G(\mathbf{k}) - \lambda I)v = 0$ by substitution. I guess Mathematica could help you too.