05/06/2018. Coherence & RIP

$y = Ax_0$ $\quad$ x_0 is k-sparse. $A \in \mathbb{R}^{m \times n}$, $m < n$

$(P_{0,1})$ $\min \|x\|_{0,1}$: $Ax = y$

Nullspace property of order k: $\forall I$, $|I| \leq k$, (NSP(k))

$$\|h_I\|_1 < \|h_{I^c}\|_1 \quad \forall h \in \text{Null}(A)$$

$\Leftrightarrow$ uniform recovery of k-sparse x_0 from (P_1) $\quad$ $(UR_1(k))$

$\Rightarrow$ uniform recovery of k-sparse x_0 from (P_0) $\quad$ $(UR_0(k))$

Prop. $(UR_0(k)) \Leftrightarrow$ Null(A) does not contain $2k$-sparse vectors, other than 0.

Pf. $(\Leftarrow)$ Say x, z are k-sparse with $y = Ax = Az$. Then $x - z$ is $2k$-sparse and $A(x-z) = 0 \Rightarrow x = z$

$(\Rightarrow)$ Say $v \in \text{Null}(A)$ is $2k$-sparse. Then $v = x - z$ with $\text{supp}\, x \cap \text{supp}\, z = \emptyset$, x, z are k-sparse, and $Ax = Az$. But $UR_0(k)$ implies $x = z$, hence $v = 0$ since disjoint support. $\square$

Cor. $A \in \mathbb{R}^{m \times n}$ needs to have $2k$ lin. indep. columns $\Rightarrow$ $m \geq 2k$ is necessary for $UR_0(k)$

(Note: $m \geq k+1$ for nonuniform recovery).

Cor. $m < 2k \implies UR_0(k)$ cannot hold
$\implies UR_1(k)$ cannot hold
$\implies NSP(k)$ cannot hold.

ex. $A = \left[\quad \boxed{A_I} \,\big|\, \big| \quad \right]$

$a_i = a_{i+1}$ → no reason to choose a_i over a_{i+1}.

Matrices favorable for $UR_{0,1}(k)$ have sufficiently different columns.

Def. (Coherence) Assume $A = [a_1 | \cdots | a_n]$ with $\|a_j\|_2 = 1$. Then

$$\mu = \max_{i \neq j} |\langle a_i, a_j \rangle| \qquad (\mu \leq 1, \text{clearly})$$

ex. $A = [I | F]$ where F is the DFT in $\mathbb{C}^m$
$(n = 2m)$

$$a_{m+j}(k) = \frac{1}{\sqrt{m}} e^{ijk/m} \qquad \mu = \frac{1}{\sqrt{m}}$$

Prop If $\mu < \frac{1}{2k-1}$, then $UR_1(k)$

Pf. STP $\left\{ \begin{array}{l} v \in \mathrm{Null}(A) \\ |I| = k \end{array} \right. \implies \|v_I\|_1 < \|v_{I^c}\|_1$

$$\sum v_j a_j = 0$$

$i \in I$, $\quad v_i = v_i \langle a_i, a_i \rangle = -\sum_{j \neq i} v_j \langle a_j, a_i \rangle$

$$|v_i| \leq \sum_{j \neq i} |v_j| \, |\langle a_j, a_i \rangle|$$

$$= \Big(\sum_{j \in I^c} + \sum_{\substack{j \in I \\ j \neq i}} \Big) |v_j| \, |\langle a_j, a_i \rangle|$$

$$\sum_{i \in I} |v_i| \leq \sum_{j \in I^c} |v_j| \sum_{i \in I} |\langle a_j, a_i \rangle|$$

$$+ \sum_{j \in I} |v_j| \sum_{\substack{i \in I \\ i \neq j}} |\langle a_j, a_i \rangle|$$

$$\|v_I\|_1 \leq \|v_{I^c}\|_1 \, k\mu + \|v_I\|_1 (k-1)\mu$$

$$\|v_I\|_1 \leq \underbrace{\frac{k\mu}{1-(k-1)\mu}}_{<1} \|v_{I^c}\|_1$$

$$< 1 \iff (2k-1)\mu < 1 \qquad \square$$

ex. $\mu = \frac{1}{\sqrt{m}}$ for union of 2 incoherent bases

$\Rightarrow UR_1(k)$ when $m > (2k-1)^2 \sim k^2$

ref: Donoho et al. 1998, 2001

Compressed sensing: stronger conditions on A
$\Rightarrow UR_1(k)$ with $m \gtrsim k \log n$

Def. (Restricted isometry property, Candès-Tao 2006)
δ_k is smallest $\delta \geq$ such that

$$(1-\delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta)\|x\|_2^2$$

$\forall$ k-sparse x (same normalization of columns as before)

$$\iff \delta_k = \max_{|I|=k} \|A_I^* A_I - Id\|$$

<u>Thm</u>. Assume $\delta_{2k} < \frac{1}{3}$. Then $UR_1(k)$.
(Candès–Tao 2006)

Pf. STP $NSP(k)$. Let $v \in Null(A)$.

$I = I_0$ = set of k largest entries of v
I_1 = next k largest entries
I_2 =
$I_0^c = I_1 \cup I_2 \cup \ldots$

$v_{(j)}$ = restriction of v on I_j.
(zero elsewhere)

$$\|v_{(0)}\|_2^2 \le \frac{1}{1-\delta_{2k}} \|Av_{(0)}\|_2^2$$

$Av = 0$
with $v = \sum_{j=0} v_{(j)}$

$$= \frac{1}{1-\delta_{2k}} \left\langle Av_{(0)}, \sum_{j=1} A(-v_{(j)}) \right\rangle$$

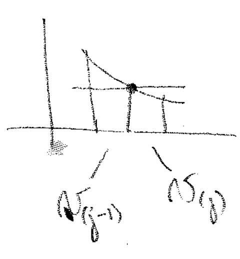
Let $S = I_0 \cup I_j$,

$$|\langle Av_{(0)}, Av_{(j)} \rangle| = |\langle A_S (v_{(0)})_S, A_S (v_{(j)})_S \rangle - \langle (v_{(0)})_S, (v_{(j)})_S \rangle|$$

$$= |\langle (v_{(0)})_S, (A_S^* A_S - I)(v_{(j)})_S \rangle|$$

$$\le \delta_{2k} \|v_{(0)}\|_2 \|v_{(j)}\|_2 .$$

$$\|v_{(0)}\|_2^2 \le \frac{\delta_{2k}}{1-\delta_{2k}} \|v_{(0)}\|_2 \sum_{j=1} \|v_{(j)}\|_2$$



$$\|v_{(j)}\|_2 \le \sqrt{k} \max_\ell |v_{(j)}(\ell)|$$
$$\le \sqrt{k} \min_\ell |v_{(j-1)}(\ell)|$$
$$\le \sqrt{k} \frac{1}{k} \|v_{(j-1)}\|_1$$

$$\|v_{(0)}\|_2 \leq \frac{\delta_{2k}}{1-\delta_{2k}} \frac{1}{\sqrt{k}} \underbrace{\left[\sum_{j=1} \|v_{(j-1)}\|_1\right]}_{\leq \|v_{I_0^c}\|_1 + \|v_{(0)}\|_1} *$$

$$\|v_{I_0}\|_1 \leq \sqrt{k}\, \|v_{(0)}\|_2 \leq \frac{\delta_{2k}}{1-2\delta_{2k}} \|v_{I_0^c}\|_1 \qquad \square$$

Tight: $\frac{4}{\sqrt{41}} \leq \delta_{2k} < \frac{1}{\sqrt{2}}$

0.6246... 0.7071...

ex Gaussian random matrices

Recall $A \in \mathbb{R}^{m \times k}$ $(m > k)$, $A_{ij} \sim N(0,1)$ iid

$$\mathbb{P}\left[\sigma_{max}\left(\frac{A}{\sqrt{m}}\right) \geq 1 + \sqrt{\frac{k}{m}} + t\right] \leq e^{-mt^2/2}$$

$$\mathbb{P}\left[\sigma_{min}\left(\frac{A}{\sqrt{m}}\right) \leq 1 - \sqrt{\frac{k}{m}} - t\right] \leq e^{-mt^2/2}.$$

$\rightarrow$ use for $A_I/\sqrt{m}$

Cor. $\delta_{2s} < \frac{4}{\sqrt{41}}$ with probability $1-\varepsilon$

when $m \geq 50\left(2k \ln\left(\frac{en}{2k}\right) + \ln(2\varepsilon^{-1})\right)$.

compare with nonuniform guarantee:

$m \geq 2k \ln\left(\frac{en}{k}\right) + \ln \varepsilon^{-1}$