05/03/18

$y = A(X_0)$, $\min \|X\|_* : y = A(X)$ (P)

Candès, Recht 2008: $A(X)$ samples m elements of X at random; X drawn from a random model where U, V are random orthogonal;
$m \geq C\, n^{1.25}\, r \log n$
r = rank(X_0)
then (P) recovers X_0 with proba $1 - O(n^{-3})$

Def. $T = \{ U W_1^T + W_2 V^T : X_0 = U \Sigma V^T \}$.
$Y_T = P_T Y = (I - P_{T^\perp}) Y$

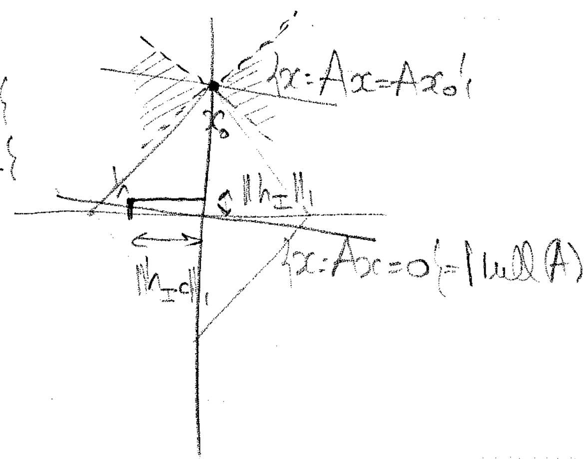
Thm. If $\exists\, Y \in \operatorname{Ran} A^*$:
(i) $Y_T = UV^T$
(ii) $\|Y_{T^\perp}\| < 1$
(iii) A_T injective
then X_0 is the unique minimizer of (P)

Pf. (see previous lecture)

5.2 Uniform recovery for $\min \|x\|_1 : Ax = y = Ax_0$ (P_1)

Let $h = x - x_0 \in Null(A)$

$I = \{1\}$
$I^c = \{2\}$



Recoverability of x_0 follows from favorable orientation of Null(A): goes through "shaded cone"

Def A obeys the cone condition / nullspace property (NSP) relative to I if $\forall h \in Null(A) \setminus 0$

$$\|h_I\|_1 < \|h_{I^c}\|_1$$

A obeys the NSP of order k if it does $\forall I : |I| = k$ $\rightarrow$ NSP(k)
(elements in the nullspace cannot be sparse)

Thm (Uniform recovery) Every k-sparse x_0 such that $Ax_0 = y$ is the unique solution of (P_1) if and only if A obeys the NSP of order k.

Pf. $\Leftarrow$ (only) Let $h = x - x_0$ (x is minimizer)

$$\begin{aligned} \|x\|_1 &= \|x_0 + h\|_1 \\ &= \|x_0 + h_I\|_1 + \|h_{I^c}\|_1 \\ &\geq \|x_0\|_1 - \|h_I\|_1 + \|h_{I^c}\|_1 \\ &> \|x_0\|_1 \quad \text{unless } h = 0. \end{aligned}$$

$\Leftrightarrow x - x_0$

This is true regardless of x_0, I □

Cor. When NSP(k) holds, then the solution of (P_1) is also the solution of (P_0):

$$\min \|x\|_{\ell_0} : Ax = y = Ax_0. \qquad (P_0)$$

Pf. Let z be a minimizer of (P_0), x_0 min (P_1)

Then $\|z\|_{\ell_0} \leq \|x_0\|_{\ell_0} = k$

$\Rightarrow$ z is k-sparse and obeys $Az = y$.

But NSP(k) $\Rightarrow$ every k-sparse z : $Az = y$ is the unique minimizer of (P_1)

$\Rightarrow$ $z = x_0$.

Credit for thm and cor: Donoho, 1998, 2001 (basis pursuit)

Robust recovery: - not exactly k-sparse
- noise.

(P_1^ε)

- $\min \|x\|_1 : \|Ax - y\| \leq \varepsilon, \qquad y = Ax_0 + e, \ \|e\| \leq \varepsilon.$
- control $\sigma_k(x_0)_1 = \inf\{\|x - x_0\|_1 : \|x\|_{\ell_0} \leq k\}$.

Def. A obeys the robust NSP of order k if $\exists\ 0 < \rho < 1,\ \tau > 0;\ \forall I : |I| = k,$

$$\|h_I\|_1 \leq \rho \|h_{I^c}\|_1 + \tau \|Ah\|_2$$

Thm (Uniform, robust recovery) Assume A obeys the robust NSP(k). Then, $\forall x_0 \in \mathbb{R}^n$, any solution x of (P_1^ε) obeys

$$\|x - x_0\|_1 \leq 2\,\frac{1+\rho}{1-\rho}\,\sigma_k(x_0)_1 + 4\,\frac{\tau}{1-\rho}\,\varepsilon$$

Pf. Let $h = x - x_0$. (I not supp x_0)

(i) $\|h_I\|_1 \le \rho \|h_{I^c}\|_1 + \tau \|Ah\|$

(ii) $\|x_0\|_1 = \|(x_0)_{I^c}\|_1 + \|(x_0)_I\|_1$
$\le \|(x_0)_{I^c}\|_1 + \|\underbrace{(x_0 - x)}_{-h}{}_I\|_1 + \|x_I\|_1$

(iii) $\|\underbrace{(x - x_0)}_{h}{}_{I^c}\|_1 \le \|x_{I^c}\|_1 + \|(x_0)_{I^c}\|_1$

Add (ii), (iii) to get

$$\|x_0\|_1 + \|h_{I^c}\|_1 \le 2\|(x_0)_{I^c}\|_1 + \|h_I\|_1 + \|x\|_1$$

$$\|h_{I^c}\|_1 \le 2\underbrace{\|(x_0)_{I^c}\|_1}_{\le \sigma_k(x_0)_1} + \underbrace{(\|x\|_1 - \|x_0\|_1)}_{\le 0} + \underbrace{\|h_I\|_1}_{\le \rho\|h_{I^c}\|_1 + \tau\|Ah\|}$$

with $\|Ah\| \le \|Ax - y\| + \|y - Ax_0\|$
$\le 2\varepsilon$

$$\|h_{I^c}\|_1 \le \frac{1}{1-\rho}\left(2\sigma_k(x_0)_1 + 2\tau\varepsilon\right). \qquad \text{(iv)}$$

$$\|h_I\|_1 \le \rho\|h_{I^c}\|_1 + \tau\|Ah\|$$
$$\le \frac{\rho}{1-\rho}(2\sigma_k(x_0)_1 + 2\tau\varepsilon) + 2\tau\varepsilon \qquad \text{(v)}$$
$$\le \frac{\rho}{1-\rho} 2\sigma_k(x_0)_1 + \frac{1}{1-\rho} 2\tau\varepsilon$$

Add (iv), (v) to get

$$\|h\|_1 \le 2\frac{1+\rho}{1-\rho}\sigma_k(x_0)_1 + 4\frac{\tau}{1-\rho}\varepsilon \qquad \square$$