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Matrix completion: FB vs SVT (Uzawa)

FB: $Y_k = X_k + \delta P_\Omega(A - X_k)$
$X_{k+1} = S_{\lambda\delta}(Y_k)$.

SVT: $Y_k = Y_{k-1} + \delta P_\Omega(A - X_k)$
$X_{k+1} = S_\lambda(Y_k)$

FB solves $\min \|X\|_* + \frac{1}{2\lambda}\|P_\Omega(A-X)\|_F^2$

SVT solves $\min \|X\|_*$ s.t. $P_\Omega(A-X)=0$.

FB has higher-rank iterates.

Chapter 5: Recovery theory, compressed sensing (CS)

5.1 Dual certification
5.2 Nullspace property & restricted isometry property (RIP)

CS question: $y = Ax_0 + e$, $\|e\| \le \sigma$, $A \in \mathbb{R}^{m\times n}$, $n > m$
Say x_0 is k-sparse.

- $x = \arg\min_x \|x\|_1$ s.t. $\|y - Ax\| \le \sigma$,

how close is x from x_0? Is $x = x_0$ when $\sigma = 0$?

→ depends on properties of A, and on k.

Dual certification

(P) $\min f_0(x) : \quad Ax = y$, say $y = Ax_0$

$\mathcal{L}(x,\lambda) = f_0(x) - \lambda^T(Ax-y)$

$f_0 \in C^1$, convex: KKT conditions

$$(1) \quad \nabla_x \mathcal{L}(x,\lambda) = \nabla f_0 - A^T\lambda = 0$$
$$(2) \quad \nabla_\lambda \mathcal{L}(x,\lambda) = Ax - y = 0$$

Q. Is x_0 a minimizer?

If
- x_0 is feasible: $Ax_0 = y$
- $\exists\, \lambda$ s.t. $A^T\lambda = \nabla f_0(x_0)$,

then yes because KKT are satisfied.
$A^T\lambda$ is a dual certificate: its existence certifies that x_0 is a minimizer.
(then λ is dual-optimal, and strong duality holds).

$f_0 \in C^0$, convex: the first KKT condition at x_0 becomes

$$(1) \quad 0 \in \partial_x \mathcal{L}(x_0,\lambda) = \partial f_0(x_0) - A^T\lambda$$
$$\Leftrightarrow \quad A^T\lambda \in \partial f_0(x_0)$$

$$\Leftrightarrow \quad f(x) \geq f(x_0) + \langle A^T\lambda, x - x_0\rangle$$
$$f(x) \geq f(x_0) + \lambda^T A(x-x_0)$$

Provides a direct proof of optimality: if $Ax = Ax_0$, then $f(x) \geq f(x_0)$.

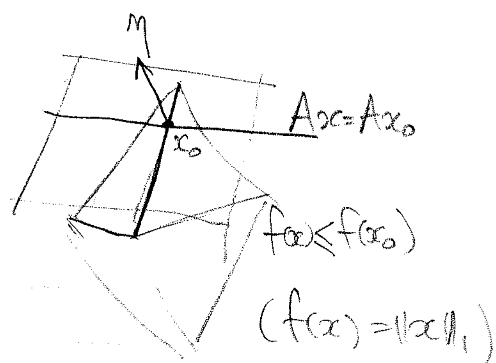
Interpretation of $\eta = A^T\lambda$: defines a codimension-1 hyperplane P_η in x-space, of equation $\eta^T(x-x_0)=0$.

(i) in half-space $\eta^T(x-x_0) > 0$,
then $f(x) \geq f(x_0) + \eta^T(x-x_0) > f(x_0)$.

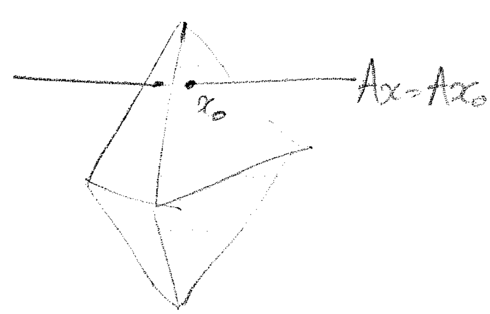
$\Rightarrow \{x : f(x) \leq f(x_0)\}$ is in the region $\eta^T(x-x_0) \leq 0$.
$\Rightarrow$ all on one side of P_η

(ii) $\eta = A^T\lambda$ for some λ

$\Rightarrow \{x : Ax = Ax_0\}$ lies in $\eta^T(x-x_0)=0$
$\Rightarrow$ all on the other side of P_η



$\rightarrow$ P_η separates the two convex sets. (and on this picture, the minimizer is unique).



$\rightarrow$ for this x_0, there is no η defining a hyperplane that separates the 2 sets. //