18.327 Computational Inverse Problems – Spring 2018
Problem set 2 – Due 03/15/2018

Problems are labeled ($\star$) for easy, ($\star\star$) for medium, and ($\star\star\star$) for hard. For homework 2, solve (at least) five stars worth of questions. Recommended exercises: 2, 4, 5, 6, 9.

1. ($\star$) Let $y \in \mathbb{R}^n$, and $B \in \mathbb{R}^{(n-2)\times n}$ be the second-difference (rectangular) matrix

$$B = \begin{pmatrix} 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \end{pmatrix}.$$

   Show that the Tikhonov-regularized problem $\min_x \|x - y\|^2 + \lambda \|Bx\|^2$, in the limit $\lambda \to \infty$, becomes least-squares linear regression.

2. ($\star$) Consider the primal problem

$$\min_x f_0(x) \quad \text{subject to} \quad Ax = y, \qquad \text{(P)}$$

   for some convex and differentiable $f_0(x)$. Relax the constraint by considering the unconstrained Lagrangian formulation

$$\min_x \phi(x), \qquad \phi(x) = f_0(x) + \alpha \|Ax - y\|^2 \qquad \text{(L)}$$

   for some $\alpha > 0$. Show how a solution of (L) can provide a dual feasible vector for (P), and show how this dual feasible vector can be used to formulate a lower bound on the optimal value of (P).

3. ($\star$) Show that, for $x \in \mathbb{R}^n$, $\|x\|_{\ell_0} = \lim_{p\to 0} \|x\|_p^p$.

4. ($\star$) Show that $\min_{x\in\mathbb{R}^n} \|x\|_1$ subject to a single linear constraint $a^T x = b$ always has a one-sparse vector as minimizer. (This is perhaps the simplest illustration of the general phenomenon that solutions of underdetermined ℓ_1 minimization problems tend to be sparse.)

5. ($\star$) Show that $\min_{a\in\mathbb{R}} \|x - a\|_1$ has the median of the vector x as a minimizer. (This is perhaps the simplest illustration of the general phenomenon that measuring the data misfit in the ℓ_1 norm tends to be robust to outliers.)

6. ($\star$) Show that the dual norm for the spectral norm is the nuclear norm.

7. ($\star\star$) Show that the nuclear norm is the so-called atomic norm for the dictionary of rank-1 normalized matrices, i.e.,

$$\|X\|_* = \inf\{\sum_i c_i \ : \ X = \sum_i c_i A_i, \ c_i > 0, \ \text{rank}(A_i) = 1, \ \|A_i\| \ = 1\}.$$

   [Hint: Maryam Fazel's PhD thesis, section 5.1.4]

8. ($\star\star$) Recall that $X \succ 0$ means positive definite, and $X \succeq 0$ means positive semi-definite.

   a) Consider a block matrix

$$M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

   with $A \succ 0$. Show that $M \succeq 0$ if and only if $S = C - B^T A^{-1} B \succeq 0$. [Hint: Boyd and Vandenberghe, p. 650, or else perform block-gaussian elimination. The matrix S is called a Schur complement.]

b) Show that the constraint $\|X\| \leq 1$ in the spectral norm can be equivalently encoded by the linear matrix inequality

$$\begin{pmatrix} I & X \\ X^T & I \end{pmatrix} \succeq 0.$$

[Hint: use the result in part a).]

9. ($\star$) Show that the "lifting trick" for the phase retrieval problem $\min_x 0 : |a_i^T x|^2 = y_i$ can be automatically derived from taking the dual of the dual. Show that strong duality holds when the primal is feasible. [Hint: the LMI $X \succeq 0$ is handled in the Lagrangian by means of a term $-\langle M, X \rangle$ with $M \succeq 0$.]

10. ($\star\star\star$) Characterize, in any way that you see fit, the convex envelope of the ℓ_0 quasi-norm restricted to the set $\|x\|_2 \leq 1$.