18.325 - Waves and Imaging
Fall 2012 - Class notes

Laurent Demanet

Draft October 2, 2012

Preface

In this text we use the symbol ($) to draw attention every time a physical assumption or simplification is made.

Chapter 1

Wave equations

1.1 Physical models

1.1.1 Acoustic waves

Acoustic waves are propagating pressure disturbances in a gas or liquid. With $p(x,t)$ the pressure fluctuation (a time-dependent scalar field) and $v(x,t)$ the particle velocity (a time-dependent vector field), the acoustic wave equations read

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\nabla p, \tag{1.1}$$

$$\frac{\partial p}{\partial t} = -\kappa_0 \nabla \cdot v. \tag{1.2}$$

The two quantities ρ_0 and κ_0 are the mass density and the bulk modulus, respectively. They are linked to the wave speed c through $\kappa_0 = \rho_0 c^2$. Initial conditions on p and v must be supplied. A forcing term may be added to the dynamic balance equation (1.1) when external forces (rather than initial conditions) create the waves.

Let us now explain how these equations are obtained from a linearization of Euler's gas dynamics equations in a uniform background medium ($). Consider the mass density ρ as a scalar field. In the inviscid case ($), conservation of momentum and mass respectively read

$$\rho(\frac{\partial v}{\partial t} + v \cdot \nabla v) = -\nabla p, \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$

An additional equation, called constitutive relation, must be added to close the system. It typically relates the pressure and the density in an algebraic way, and encodes a thermodynamic assumption about compression and dilation. For instance if the gas is assumed to be ideal, and if the compression-dilation process occurring in the wave is adiabatic reversible (no heat transfer), then $p \sim \rho^\gamma$, $\gamma = 1.4$, where $\sim$ indicates equality up to a dimensional constant. More generally, assume for the moment that the constitutive relation takes the form

$$p = f(\rho)$$

for some scalar function f, which we assume differentiable and strictly increasing ($f'(\rho) > 0$ for all $\rho > 0$).

Consider small disturbances off of an equilibrium state:

$$p = p_0 + p_1, \qquad \rho = \rho_0 + \rho_1, \qquad v = v_0 + v_1.$$

In what follows, neglect quadratic quantities of p_1, ρ_1, v_1. Consider a medium at rest (\$): p_0, ρ_0 independent of t, and $v_0 = 0$. After some algebraic simplification the conservation of momentum becomes

$$\rho_0 \frac{\partial v_1}{\partial t} = -\nabla p_0 - \nabla p_1.$$

To zero-th order (i.e., at equilibrium, $p_1 = \rho_1 = v_1 = 0$,) we have

$$\nabla p_0 = 0 \qquad \Rightarrow \qquad p_0 \text{ constant in } x.$$

To first order, we get

$$\rho_0 \frac{\partial v_1}{\partial t} = -\nabla p_1,$$

which is exactly (1.1) after renaming $v_1 \to v$, $p_1 \to p$. The constitutive relation must hold at equilibrium, hence p_0 constant in x implies that ρ_0 is also constant in x (uniform). Conservation of mass becomes

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot v_1 = 0.$$

Differentiate the constitutive relation to obtain $p_1 = f'(\rho_0)\rho_1$. Call $f'(\rho_0) = c^2$, a number that we assume positive. Then we can eliminate ρ_1 to get

$$\frac{\partial p_1}{\partial t} + \rho_0 c^2 \nabla \cdot v_1 = 0.$$

This is exactly (1.2) with $\kappa_0 = \rho_0 c^2$.

Conveniently, the equations for acoustic waves in a variable medium $\rho_0(x)$, $\kappa_0(x)$ are obvious modifications of (1.1), (1.2):

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0(x)} \nabla p, \tag{1.3}$$

$$\frac{\partial p}{\partial t} = -\kappa_0(x) \nabla \cdot v. \tag{1.4}$$

A different argument is needed to justify these equations, however. The previous reasoning does not leave room for variable $\rho_0(x)$ or $\kappa_0(x)$. Instead, it is necessary to introduce a more realistic constitutive relation

$$p = f(\rho, s),$$

where s is the entropy. An additional equation for conservation of entropy needs to be considered. The new constitutive relation allows ρ_0 and s_0 to be functions of x in tandem, although p_0 is still (necessarily) uniform in x. The reasoning leading to (1.3), (1.4) is the subject of an exercise in section 1.3.

Acoustic waves can take the form of a first-order system of equations, or else a second-order scalar equation. Combining (1.3), (1.4), we get

$$\frac{\partial^2 p}{\partial t^2} = \kappa_0(x) \nabla \cdot (\frac{1}{\rho_0(x)} \nabla p).$$

Initial conditions on both p and $\partial p / \partial t$ must be supplied. This equation may come with a right-hand side $f(x,t)$ that indicates forcing. When ρ_0 and κ_0 are constant, the scalar wave equation reduces to

$$\frac{\partial^2 p}{\partial t^2} = c_0^2 \, \Delta p.$$

Waves governed by (1.3), (1.4) belong in the category of hyperbolic waves because they obey conservation of energy. Define

$$w = \begin{pmatrix} v \\ p \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & -\frac{1}{\rho_0}\nabla \\ -\kappa_0 \nabla \cdot & 0 \end{pmatrix}.$$

Then the acoustic system simply reads

$$\frac{\partial w}{\partial t} = Lw.$$

L is called the generator of the evolution.

Definition 1. *The system* $\frac{\partial w}{\partial t} = Lw$ *is said to be hyperbolic if* L *is a matrix of first-order differential operators, and there exists an inner product* $\langle w, w' \rangle$ *with respect to which* $L^* = -L$*, i.e.,* L *is anti-self-adjoint.*

An adjoint operator such as L^* is defined through the equation[1]

$$\langle Lw, w' \rangle = \langle w, L^* w' \rangle, \qquad \text{for all } w, w'.$$

For instance, in the case of the acoustic system, the proper notion of inner product is (the factor 1/2 is optional)

$$\langle w, w' \rangle = \frac{1}{2} \int (\rho_0 v \cdot v' + \frac{1}{\kappa_0} p p')\, dx.$$

It is an exercise in section 1.3 to show that $\langle Lw, w' \rangle = \langle w, L^* w' \rangle$ for that inner product, for all w, w'.

Theorem 1. *If* $\frac{\partial w}{\partial t} = Lw$ *is a hyperbolic system, then* $E = \langle w, w \rangle$ *is conserved in time.*

Proof.

$$\begin{aligned}
\frac{d}{dt} \langle w, w \rangle &= \langle \frac{\partial w}{\partial t}, w \rangle + \langle w, \frac{\partial w}{\partial t} \rangle \\
&= 2\, \langle \frac{\partial w}{\partial t}, w \rangle \\
&= 2\, \langle Lw, w \rangle \\
&= 2\, \langle w, L^* w \rangle \\
&= 2\, \langle w, (-L) w \rangle \\
&= -2\, \langle Lw, w \rangle.
\end{aligned}$$

A quantity is equal to minus itself if and only if it is zero. □

In the case of acoustic waves,

$$E = \frac{1}{2} \int (\rho_0 v^2 + \frac{p^2}{\kappa})\, dx,$$

[1]The existence of L^* can be traced back to the Riesz representation theorem once $\langle Lw, w' \rangle$ is shown to be a continuous functional of w in some adequate Hilbert space norm.

which can be understood as kinetic plus potential energy. We now see that the factor 1/2 was chosen to be consistent with the physicists' convention for energy.

In the presence of external forcings the hyperbolic system reads $\partial w/\partial t = Lw + f$: in that case the rate of change of energy is determined by f.

For reference, common boundary conditions for acoustic waves include

- Sound soft boundary condition: Dirichlet for the pressure, $p = 0$.
- Sound-hard boundary condition: Neumann for the pressure, $\frac{\partial p}{\partial n} = 0$, or equivalently $v \cdot n = 0$.

Another important physical quantity is related to acoustic waves: the acoustic impedance $Z = \sqrt{\rho_0 \kappa_0}$. We will see later that impedance jumps determine reflection and transmission coefficients at medium discontinuities.

1.1.2 Elastic waves

Elastic waves are propagating pressure disturbances in solids. The interesting physical variables are

- The displacement $u(x,t)$, a time-dependent vector field. In terms of u, the particle velocity is $v = \frac{\partial u}{\partial t}$.
- The strain tensor
  $$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T),$$
  a symmetric time-dependent tensor field.
- The stress tensor σ, also a symmetric time-dependent tensor field.

For elastic waves, the density ρ is very often assumed independent of t along particle trajectories, namely $\rho_0(x,0) = \rho_0(x + u(x,t), t)$.

The equation of elastic waves in an isotropic medium (where all the waves travel at the same speed regardless of the direction in which they propagate) ($) reads

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla(\lambda \nabla \cdot u) + \nabla \cdot (\mu(\nabla u + (\nabla u)^T)). \tag{1.5}$$

where ρ, λ, and μ may possibly depend on x. As for acoustic waves, a forcing term is added to this equation when waves are generated from external forces.

To justify this equation, start by considering the equation of conservation of momentum (“$F = ma$”),

$$\rho \frac{\partial v}{\partial t} = \nabla \cdot \sigma,$$

possibly with an additional term $f(x,t)$ modeling external forces. The notation $\nabla \cdot$ indicates tensor divergence, namely $(\nabla \cdot \sigma)_i = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j}$. Stress and strain are linked by a constitutive relation called Hooke's law,

$$\sigma = C : \epsilon,$$

where C is the 4-index elastic tensor. In three spatial dimensions, C has 81 components. The colon indicates tensor contraction, so that $(C : \epsilon)_{ij} = \sum_{k\ell} C_{ijk\ell} \epsilon_{k\ell}$.

These equations form a closed system when they are complemented by

$$\frac{\partial \epsilon}{\partial t} = \frac{1}{2}(\nabla v + (\nabla v)^T),$$

which holds by definition of ϵ.

At this point we can check that the first-order system for v and ϵ defined by the equations above is hyperbolic. Define

$$w = \begin{pmatrix} v \\ \epsilon \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & L_2 \\ L_1 & 0 \end{pmatrix},$$

with

$$L_1 v = \frac{1}{2}(\nabla v + (\nabla v)^T), \qquad L_2 \epsilon = \frac{1}{\rho_0} \nabla \cdot (C : \epsilon).$$

Then, as previously, $\frac{\partial w}{\partial t} = Lw$. An exercise in section 1.3 asks to show that the matrix operator L is anti-selfadjoint with respect to the inner product

$$\langle w, w' \rangle = \frac{1}{2} \int (\rho v \cdot v' + \epsilon : C : \epsilon) \, dx.$$

The corresponding conserved elastic energy is $E = \langle w, w \rangle$.

Isotropic elasticity is obtained where C takes a special form with 2 degrees of freedom rather than 81, namely

$$C_{ijk\ell} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{i\ell}\delta_{jk} + \delta_{ik}\delta_{j\ell}).$$

We are not delving into the justification of this equation. The two elastic parameters λ and μ are also called Lamé parameters:

- λ corresponds to longitudinal waves, also known as compressional, pressure waves (P).
- μ corresponds to transverse waves, also known as shear waves (S).

Originally, the denominations P and S come from "primary" and "secondary", as P waves tend to propagate faster, hence arrive earlier, than S waves.

With this parametrization of C, it is easy to check that the elastic system reduces to the single equation (1.5). In index notation, it reads

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_i(\lambda \partial_j u_j) + \partial_j(\mu(\partial_i u_j + \partial_j u_i)).$$

For reference, the hyperbolic propagator L_2 reduces to

$$L_2 \epsilon = \frac{1}{\rho}(\nabla(\lambda \operatorname{tr} \epsilon) + 2 \nabla \cdot (\mu \epsilon)), \qquad \operatorname{tr} \epsilon = \sum_i \epsilon_{ii},$$

and the energy inner product is

$$\langle w, w' \rangle = \frac{1}{2} \int (\rho v \cdot v' + 2 \mu \operatorname{tr}(\epsilon^T \epsilon') + \lambda (\operatorname{tr} \epsilon)(\operatorname{tr} \epsilon')) \, dx.$$

The elastic wave equation looks like an acoustic wave equation with "2 terms, hence 2 waves". To make this observation more precise, assume that λ and μ are constant (\$). Use some vector identities[2] to reduce (1.5) to

$$\begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} &= (\lambda + \mu) \nabla(\nabla \cdot u) + \mu \Delta u, \\ &= (\lambda + 2\mu) \nabla(\nabla \cdot u) - \mu \nabla \times \nabla \times u. \end{aligned}$$

Perform the Helmholtz (a.k.a. Hodge) decomposition of u in terms of potentials ϕ and ψ:

$$u = \nabla \phi + \nabla \times \psi,$$

where ϕ is a scalar field and ψ is a vector field[3]. These two potentials are determined up to a gauge choice, namely

$$\phi' = \phi + C, \qquad \psi' = \psi + \nabla f.$$

[2] In this section, we make use of $\nabla \times \nabla \times u = \nabla(\nabla \cdot u) - \Delta u$, $\nabla \cdot \nabla \times \psi = 0$, and $\nabla \times \nabla \psi = 0$.

[3] Normally the Helmholtz decomposition comes with a third term h which obeys $\Delta h = 0$, i.e., h is harmonic, but under suitable assumptions of decay at infinity the only solution to $\Delta h = 0$ is $h = 0$.

Choose f such that ψ' has zero divergence:

$$\nabla \cdot \psi' = 0 \qquad \Rightarrow \qquad \Delta f = -\nabla \cdot \psi.$$

This is a well-posed Poisson equation for f. With this choice of ψ', it holds that

$$\nabla \cdot u = \Delta \phi, \qquad \nabla \times u = \nabla \times \nabla \times u = -\Delta \psi.$$

The elastic wave equation can then be rewritten in terms of ϕ, ψ as

$$\nabla \left[\rho \frac{\partial^2 \phi}{\partial t^2} - (\lambda + 2\mu)\Delta\phi \right] + \nabla \times \left[\rho \frac{\partial^2 \psi}{\partial t^2} - \mu \Delta \psi \right] = 0.$$

Take the gradient of this equation to conclude that (with a suitable decay condition at infinity)

$$\rho \frac{\partial^2 \phi}{\partial t^2} - (\lambda + 2\mu)\Delta\phi = \text{ harmonic } = 0.$$

Now that the first term is zero, we get (with a suitable decay condition at infinity)

$$\rho \frac{\partial^2 \psi}{\partial t^2} - \mu \Delta \psi = \nabla(\text{something}) = 0.$$

Hence each potential ϕ and ψ solve their own scalar wave equation: one for the longitudinal waves (ϕ) and one for the transverse waves (ϕ). They obey a superposition principle. The two corresponding wave speeds are

$$c_P = \sqrt{\frac{\lambda + 2\mu}{\rho_0}}, \qquad c_S = \sqrt{\frac{\mu}{\rho_0}}.$$

In the limit $\mu \to 0$, we see that only the longitudinal wave remains, and λ reduces to the bulk modulus. In all cases, since $\lambda \geq 0$ we always have $c_P \geq \sqrt{2} c_S$: the P waves are indeed always faster (by a factor at least $\sqrt{2}$) than the S waves.

The assumption that λ and μ are constant is a very strong one: there is a lot of physics in the coupling of ϕ and ψ that the reasoning above does not capture. Most important is mode conversion as a result of wave reflection at discontinuity interfaces of $\lambda(x)$ and/or $\mu(x)$.

1.1.3 Electromagnetic waves

The quantities of interest for electromagnetic waves are:

- Physical fields: the electric field E, and the magnetic field H,
- Medium parameters: the electric permittivity ϵ and the magnetic permeability μ,
- Forcings: electric currents j and electric charges ρ.

The electric displacement field D and the magnetic induction field B are also considered. In the linearized regime ($), they are assumed to be linked to the usual fields E and H by the constitutive relations

$$D = \epsilon E, \qquad B = \mu H.$$

Maxwell's equations in a medium with possible space-varying parameters ϵ and μ read

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \text{(Faraday's law)} \tag{1.6}$$

$$\nabla \times H = \frac{\partial D}{\partial t} + j \qquad \text{(Ampère's law with Maxwell's correction)} \tag{1.7}$$

$$\nabla \cdot D = \rho \qquad \text{(Gauss's law for the electric field)} \tag{1.8}$$

$$\nabla \cdot B = 0 \qquad \text{(Gauss's law for the magnetic field)} \tag{1.9}$$

The integral forms of these equations are obtained by a volume integral, followed by a reduction to surface equations by Stokes's theorem for (1.6), (1.7) and the divergence (Gauss's) theorem for (1.8), (1.9). The integral equations are valid when ϵ and μ are discontinuous, whereas the differential equations strictly speaking are not.

The total charge in a volume V is $\int_V \rho dV$, while the total current through a surface S is $\int_S j \cdot dS$. Conservation of charge follows by taking the divergence of (1.7) and using (1.8):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0.$$

In vacuum, or dry air, the parameters are constant and denoted $\epsilon = \epsilon_0$, $\mu = \mu_0$. They have specific numerical values in adequate units.

We now take the viewpoint that (1.6) and (1.7) are evolution equations for E and H (or D and B) that fully determine the fields when they are solved forward (or backward) in time. In that setting, the other two equations (1.8) and (1.9) are simply constraints on the initial (or final) condition at $t = 0$. As previously, we may write Maxwell's equations in the more concise hyperbolic form

$$\frac{\partial w}{\partial t} = Lw + \begin{pmatrix} -j/\epsilon \\ 0 \end{pmatrix}, \qquad \text{with } w = \begin{pmatrix} E \\ H \end{pmatrix},$$

provided

$$L = \begin{pmatrix} 0 & \frac{1}{\epsilon}\nabla\times \\ -\frac{1}{\mu}\nabla\times & 0 \end{pmatrix}.$$

The "physical" inner product that makes $L^* = -L$ is

$$\langle w, w' \rangle = \frac{1}{2}\int (\epsilon E E' + \mu H H')\, dx.$$

The electromagnetic energy $E = \langle w, w \rangle$ is conserved when $j = 0$.

It is the balanced coupling of E and H through (1.6) and (1.7) that creates wave-like solutions to Maxwell's equations (and prompts calling the physical phenomenon electromagnetism rather than just electricity and magnetism.) Combining both equations, we obtain

$$\frac{\partial^2 E}{\partial t^2} = -\frac{1}{\epsilon}\nabla \times (\frac{1}{\mu}\nabla \times E),$$

$$\frac{\partial^2 H}{\partial t^2} = -\frac{1}{\mu}\nabla \times (\frac{1}{\epsilon}\nabla \times H).$$

These wave equations may be stand-alone but E and H are still subject to essential couplings.

A bit of algebra[4] reveals the more familiar form

$$\Delta E - \epsilon\mu\frac{\partial^2 E}{\partial t^2} + \frac{\nabla\mu}{\mu} \times (\nabla \times E) + \nabla(E \cdot \frac{\nabla\epsilon}{\epsilon}) = 0.$$

We now see that in a uniform medium, ϵ and μ are constant and the last two terms drop, revealing a wave equation with speed

$$c = \frac{1}{\sqrt{\epsilon\mu}}.$$

[4]Using the relations $\nabla \times \nabla \times F = \nabla(\nabla \cdot F) - \Delta F$ again, as well as $\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$.

The speed of light is $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$. Even when ϵ and μ vary in x, the last two terms are kinematically much less important than the first two because they involve lower-order derivatives of E. They would not, for instance, change the path of the "light rays", a concept that we'll make clear later.

For reference, we now list the jump conditions that the electric and magnetic fields obey at a dielectric interface. These relations can be obtained from the integral form of Maxwell's equations, posed over a thin volume straddling the interface. Let n be the vector normal to a dielectric interface.

$$n \times E_1 = n \times E_2 \qquad \text{(continuous tangential components)}$$

$$n \times H_1 = n \times H_2 + j_S$$

$$n \cdot D_1 = n \cdot D_2 + \rho_S$$

$$n \cdot H_1 = n \cdot H_2 \qquad \text{(continuous normal component)}$$

We have used j_S and ρ_S for surface currents and surface charges respectively. If the two dielectrics correspond to finite parameters ϵ_1, ϵ_2 and μ_1, μ_2, then these currents are zero. If material 2 is a perfect electric conductor however, then these currents are not zero, but the fields E_2, H_2, D_2 and H_2 are zero. This results in the conditions $n \times E = 0$ (E perpendicular to the interface) and $n \times H = 0$ (H parallel to the interface) in the vicinity of a perfect conductor.

Materials conducting current are best described by a complex electric permittivity $\epsilon = \epsilon' + i\sigma/\omega$, where σ is called the conductivity. All these quantities could be frequency-dependent. It is the ratio σ/ϵ' that tends to infinity when the conductor is "perfect". Materials for which ϵ is real are called "perfect dielectrics": no conduction occurs and the material behaves like a capacitor. We will only consider perfect dielectrics in this class. When conduction is present, loss is also present, and electromagnetic waves tend to be inhibited. Notice that the imaginary part of the permittivity is σ/ω, and not just σ, because we want Ampère's law to reduce to $j = \sigma E$ (the differential version of Ohm's law) in the time-harmonic case and when $B = 0$.

1.2 Special solutions

1.2.1 Plane waves, dispersion relations

In this section we study special solutions of wave equations that depend on x like e^{ikx}. These solutions are obtained if we assume that the time dependence is harmonic, namely if the unknown is $w(x,t)$, then we assume ($)

$$w(x,t) = e^{-i\omega t} f_\omega(x), \qquad \omega \in \mathbb{R}.$$

The number ω is called angular frequency, or simply frequency. Choosing $e^{+i\omega t}$ instead makes no difference down the road. Under the time-harmonic assumption, the evolution problem $\frac{\partial w}{\partial t} = Lw$ becomes an eigenvalue problem:

$$-i\omega f_\omega = L f_\omega.$$

Not all solutions are time-harmonic, but all solutions are *superpositions* of harmonic waves at different frequencies ω. Indeed, if $w(x,t)$ is a solution, consider it as the inverse Fourier transform of some $\hat{w}(x,\omega)$:

$$w(x,t) = \frac{1}{2\pi} \int e^{-i\omega t} \hat{w}(x,\omega) d\omega.$$

Then each $\hat{w}(x,\omega)$ is what we called $f_\omega(x)$ above. Hence there is no loss of generality in considering time-harmonic solutions.

Consider the following examples.

- The one-way, one-dimensional wave equation

  $$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \qquad x \in \mathbb{R}.$$

  Time harmonic solutions $u(x,t) = e^{-i\omega t} f_\omega(x)$ obey

  $$i\frac{\omega}{c} f_\omega = f_\omega', \qquad x \in \mathbb{R}.$$

  The solution to this equation is

  $$f_\omega(x) = e^{ikx}, \qquad k = \frac{\omega}{c} \in \mathbb{R}.$$

  Evanescent waves corresponding to decaying exponentials in x and t are also solutions over a half-line, say, but they are ruled out by our assumption ($) that $\omega \in \mathbb{R}$.

  While ω is the angular frequency (equal to $2\pi/T$ where T is the period), k is called the wave number (equal to $2\pi/\lambda$ where λ is the wavelength.) It is like a "spatial frequency", though it is prudent to reserve the word frequency for the variable dual to time. The quantity measured in Hertz [1/s] and also called frequency is $\nu = \omega/(2\pi)$.

The full solution then takes the form

$$u(x,t) = e^{i(kx-\omega t)} = e^{ik(x-ct)},$$

manifestly a right-going wave at speed c. If the equation had been $\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0$ instead, the wave would have been left-going: $u(x,t) = e^{ik(x+ct)}$.

- The n-dimensional wave equation in a uniform medium,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u, \qquad x \in \mathbb{R}^n.$$

When $u(x,t) = e^{-i\omega t} f_\omega(x)$, the eigenvalue problem is called the (homogeneous) Helmholtz equation. It is

$$-\omega^2 f_\omega(x) = \Delta f_\omega(x), \qquad x \in \mathbb{R}^n. \tag{1.10}$$

Again, plane waves are solutions to this equation:

$$f_\omega(x) = e^{ik\cdot x},$$

provided $\omega^2 = |k|^2 c^2$, i.e., $\omega = \pm |k| c$. Hence f_ω is a function that oscillates in the direction parallel to k. The full solution is

$$u(x,t) = e^{i(k\cdot x - \omega t)},$$

which are plane waves traveling with speed c, along the direction k. We call k the *wave vector* and $|k|$ the *wave number*. The wavelength is still $2\pi/|k|$. The relation $\omega^2 = |k|^2 c^2$ linking ω and k, and encoding the fact that the waves travel with velocity c, is called the *dispersion relation* of the wave equation.

Note that $e^{ik\cdot x}$ are not the only (non-growing) solutions of the Helmholtz equation in free space; so is any linear combination of $e^{ik\cdot x}$ that share the same wave vector $|k|$. This superposition can be a discrete sum or a continuous integral. An exercise in section 1.3 deals with the continuous superposition with constant weight of all the plane waves with same wave vector $|k|$.

Consider now the general case of a hyperbolic system $\frac{\partial w}{\partial t} = Lw$, with $L^* = -L$. The eigenvalue problem is $-i\omega f_\omega = Lf_\omega$. It is fine to assume ω real: since L is antiselfadjoint, iL is selfadjoint (Hermitian), hence all the eigenvalues of L are purely imaginary. This is sometimes how hyperbolic systems are defined — by assuming that the eigenvalues of the generator L are purely imaginary.

We still look for eigenfunctions with a $e^{ik\cdot x}$ dependence, but since w and f_ω may now be vectors with m components, we should make sure to consider

$$f_\omega(x) = e^{ik\cdot x} r, \qquad r \in \mathbb{R}^m.$$

However, such f_ω cannot in general expected to be eigenvectors of L. It is only when the equation is *translation-invariant* that they will be. This means that the generator L is a matrix of differential operators with constant coefficients – no variability as a function of x is allowed. In this translation-invariant setting, and only in this setting, L is written as a multiplication by some matrix $P(k)$ in the Fourier domain. Say that f has m components $(f_1, \ldots f_m)$; then

$$Lf(x) = \frac{1}{(2\pi)^n} \int e^{ik\cdot x} P(k) \hat{f}(k) dk,$$

where $P(k)$ is an m-by-m matrix for each k. Here $P(k)$ is called the dispersion matrix. We operators such as L *diagonal* in the Fourier domain, with respect to the k variable, because they act like a "diagonal matrix" on vectors of the continuous index k — although for each k the small matrix $P(k)$ is not in general diagonal[5]. In pure math, $P(k)$ is called the multiplier, and L is said to be a multiplication operator in the Fourier domain.

For illustration, let us specialize our equations to the 2D acoustic system with $\rho_0 = \kappa_0 = c = 1$, where

$$w = \begin{pmatrix} v \\ p \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 & -\frac{\partial}{\partial x_1} \\ 0 & 0 & -\frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & 0 \end{pmatrix}.$$

[5]Non-diagonal, translation-variant operators would require yet another integral over a k' variable, and would read $Lf(x) = \frac{1}{(2\pi)^n} \int\int e^{ik\cdot x} Q(k,k') \hat{f}(k') dk'$, for some more complicated object $Q(k,k') \in \mathbb{R}^{m\times m}$. The name "diagonal" comes from the fact that $Q(k,k')$ simplifies as $P(k)\delta(k-k')$ in the translation-invariant case. You can think of $P(k)\delta(k-k')$ as the continuous analogue of $d_i\delta_{ij}$: it is a "diagonal continuous matrix" as a function of k (continuous row index) and k' (continuous column index).

It can be readily checked that

$$P(k) = \begin{pmatrix} 0 & 0 & -ik_1 \\ 0 & 0 & -ik_2 \\ \\ -ik_1 & -ik_2 & 0 \end{pmatrix},$$

from which it is apparent that $P(k)$ is a skew-Hermitian matrix: $P^*(k) = -P(k)$.

We can now study the conditions under which $-i\omega f_\omega = L f_\omega$: we compute (recall that r is a fixed vector)

$$\begin{aligned} L(e^{ik\cdot x} r) &= \frac{1}{(2\pi)^n} \int e^{ik'\cdot x} P(k') \widehat{[e^{ik\cdot x} r]}(k') dk', \\ &= \frac{1}{(2\pi)^n} \int e^{ik'\cdot x} P(k') (2\pi)^n \delta(k-k') r dk', \qquad = e^{ik\cdot x} P(k) r. \end{aligned}$$

In order for this quantity to equal $-i\omega e^{ik\cdot x} r$ for all x, we require (at $x = 0$)

$$P(k)\, r = -i\omega\, r.$$

This is just the condition that $-i\omega$ is an eigenvalue of $P(k)$, with eigenvector r. We should expect both ω and r to depend on k. For instance, in the 2D acoustic case, the eigen-decomposition of $P(k)$ is

$$\lambda_0(k) = -i\omega_0(k) = 0, \qquad r_0(k) = \begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix}$$

and

$$\lambda_\pm(k) = -i\omega_\pm(k) = -i|k|, \qquad r_\pm(k) = \begin{pmatrix} \pm k_1/|k| \\ \pm k_2/|k| \\ |k| \end{pmatrix}.$$

Only the last two eigenvalues correspond to physical waves: they lead to the usual dispersion relations $\omega(k) = \pm|k|$ in the case $c = 1$. Recall that the first two components of r are particle velocity components: the form of the eigenvector indicates that those components are aligned with the direction k of the wave, i.e., acoustic waves can only be longitudinal.

The general definition of dispersion relation follows this line of reasoning: there exists one dispersion relation for each eigenvalue λ_j of $P(k)$, and $-i\omega_j(k) = \lambda_j(k)$; for short

$$\det\left[i\omega I + P(k)\right] = 0.$$

1.2.2 Traveling waves, characteristic equations

We now consider a few examples that build up to the notion of characteristic curve/surface.

- Let us give a complete solution to the one-way wave equation of one space variable in a uniform medium:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = u_0(x). \tag{1.11}$$

The study of plane wave solutions in the previous section suggests that the variable $x - ct$ may play a role. Let us perform the change of variables

$$\xi = x - ct, \qquad \eta = x + ct.$$

It inverts as

$$x = \frac{\xi + \eta}{2}, \qquad t = \frac{\eta - \xi}{2c}.$$

By the chain rule, e.g.,

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t},$$

we get

$$-2c\frac{\partial}{\partial \xi} = \frac{\partial}{\partial t} - c\frac{\partial}{\partial x}, \qquad 2c\frac{\partial}{\partial \eta} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}.$$

With $U(\xi,\eta) = u(x,t)$, the wave equation simply becomes

$$\frac{\partial U}{\partial \eta} = 0,$$

whose general solution is $U(\xi,\eta) = F(\xi)$ for some differentiable function F. Hence $u(x,t) = F(x - ct)$. In view of the initial condition, this is

$$u(x,t) = u_0(x - ct).$$

The solutions to (1.11) are all the right-going waves with speed c, and nothing else.

The wave propagate along the lines $\xi(x,t) = x - ct =$const. in the (x,t) plane. For this reason, we call ξ the *characteristic coordinate*, and we call the lines $\xi(x,t) =$ const. *characteristic curves.*

Notice that imposing a boundary condition $u(0,t) = v_0(t)$ rather than an initial condition is also fine, and would result in a solution $u(x,t) = v_0(t - x/c)$. Other choices are possible; they are called Cauchy data. However, a problem occurs if we try to specify Cauchy data along a characteristic curve $\xi =$ constant, as $v_0(\eta)$:

1. this choice is not in general compatible with the property that the solution should be constant along the characteristic curves; and furthermore
2. it fails to determine the solution away from the characteristic curve.

In other words, there is a problem with both existence and uniqueness when we try to prescribe Cauchy data on a characteristic curve. This fact will be used in the sequel to define these curves when their geometric intuition becomes less clear.

- Using similar ideas, let us describe the full solution of the (two-way) wave equation in one space dimension,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x).$$

The same change of variables leads to the equation

$$\frac{\partial U}{\partial \xi \partial \eta} = 0,$$

which is solved via

$$\frac{\partial U}{\partial \eta}(\xi,\eta) = f(\xi), \qquad U(\xi,\eta) = \int^{\xi} f(\xi')d\xi' + G(\eta) = F(\xi) + G(\eta).$$

The resulting general solution is a superposition of a left-going wave and a right-going wave:

$$u(x,t) = F(x - ct) + G(x + ct).$$

Matching the initial conditions yields d'Alembert's formula (1746):

$$u(x,t) = \frac{1}{2}(u_0(x - ct) + u_0(x + ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)dy.$$

It is the complete solution to the 1D wave equation in a uniform wave speed c. Notice that we now have two families of criss-crossing characteristic curves, given by $\xi(x,t) = \text{const.}$ and $\eta(x,t) = \text{const.}$ Cauchy data cannot be prescribed on either type of characteristics.

- Consider now the wave equation in a variable medium $c(x)$ (technically, acoustic waves on an infinite string with variable bulk modulus):

$$\frac{\partial^2 u}{\partial t^2} - c^2(x)\frac{\partial^2 u}{\partial x^2} = 0, \qquad u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x).$$

We will no longer be able to give an explicit solution to this problem, but the notion of characteristic curve remains very relevant. Consider an as-yet-undetermined change of coordinates $(x,t) \mapsto (\xi, \eta)$, which generically changes the wave equation into

$$\alpha(x)\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \xi \partial \eta} + \beta(x)\frac{\partial^2 U}{\partial \eta^2} + \left[p(x)\frac{\partial U}{\partial \xi} + q(x)\frac{\partial U}{\partial \eta} + r(x)U \right] = 0,$$

with

$$\alpha(x) = \left(\frac{\partial \xi}{\partial t}\right)^2 - c^2(x)\left(\frac{\partial \xi}{\partial x}\right)^2,$$

$$\beta(x) = \left(\frac{\partial \eta}{\partial t}\right)^2 - c^2(x)\left(\frac{\partial \eta}{\partial x}\right)^2.$$

The lower-order terms in the square brackets are kinematically less important than the first three terms[6]. We wish to define characteristic coordinates as those along which

$$U(\xi, \eta) \simeq F(\xi) + G(\eta),$$

i.e., "directions in which the waves travel" in space-time. It is in general impossible to turn this approximate equality into an actual equality (because of the terms in the square brackets), but it is certainly possible to choose the characteristic coordinates so that the $\frac{\partial^2 U}{\partial \xi^2}$ and $\frac{\partial^2 U}{\partial \eta^2}$ vanish. Choosing $\alpha(x) = \beta(x) = 0$ yields the same equation for both ξ and η, here expressed in terms of ξ:

$$\left(\frac{\partial \xi}{\partial t}\right)^2 - c^2(x)\left(\frac{\partial \xi}{\partial x}\right)^2 = 0. \tag{1.12}$$

[6] In a sense that we are not yet ready to make precise. Qualitatively, they affect the shape of the wave, but not the character that the waves travel with local speed $c(x)$.

This relation is called the *characteristic equation*. Notice that $\xi = x - ct$ and $\eta = x + ct$ are both solutions to this equation in the case when $c(x) = c$ is a constant. But it can be checked that $\xi = x \pm c(x)t$ is otherwise not a solution of (1.12). Instead, refer to the exercise section for a class of solutions to (1.12).

- Consider now the n dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2(x)\Delta u = 0, \qquad u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x).$$

A change of variables would now read $(x_1, \ldots, x_n, t) \mapsto (\xi, \eta_1, \ldots, \eta_n)$. The variable ξ is called characteristic when the coefficient of the leading term $\frac{\partial^2 U}{\partial \xi^2}$ vanishes in the expression of the equation in the new coordinates. This condition leads to the n-dimensional version of the characteristic equation

$$\left(\frac{\partial \xi}{\partial t}\right)^2 - c^2(x)|\nabla_x \xi|^2 = 0. \tag{1.13}$$

The same relations should hold for the other coordinates $\eta_1, \ldots, \eta_n$ if they are to be characteristic as well. Equation (1.13) is called a *Hamilton-Jacobi* equation. We now speak of characteristic surfaces $\xi(x,t) = \text{const.}$, rather than curves.

The set of solutions to (1.13) is very large. In the case of constant c, we can check that possible solutions are

$$\xi(x,t) = x \cdot k \pm \omega t, \qquad \omega = |k|c,$$

corresponding to more general plane waves $u(x,t) = F(x \cdot k \pm \omega t)$ (which the reader can check are indeed solutions of the n-dimensional wave equation for smooth F), and

$$\xi(x,t) = \|x - y\| \pm ct, \qquad \text{for some fixed } y, \text{ and } x \neq y,$$

corresponding to concentric spherical waves originating from y. We describe spherical waves in more details in the next section. Notice that both formulas for ξ reduce in some sense to $x \pm ct$ in the one-dimensional case.

The choice of characteristic coordinates led to the reduced equation

$$\frac{\partial^2 U}{\partial \xi \partial \eta} + \text{ lower order terms } = 0,$$

sometimes called "first fundamental form" of the wave equation, on the intuitive basis that solutions (approximately) of the form $F(\xi) + G(\eta)$ should travel along the curves $\xi = $ const. and $\eta = $ const. Let us now motivate this choice of the reduced equation in more precise terms, by linking it to the idea that *Cauchy data cannot be prescribed on a characteristic curve.*

Consider $u_{tt} = c^2 u_{xx}$. Prescribing initial conditions $u(x, 0) = u_0$, $u_t(x, 0) = u_1$ is perfectly acceptable, as this completely and uniquely determines all the partial derivatives of u at $t = 0$. Indeed, u is specified through u_0, and all its x-partials $u_x, u_{xx}, u_{xxx}, \ldots$ are obtained from the x-partials of u_0. The first time derivative u_t at $t = 0$ is obtained from u_1, and so are $u_{tx}, u_{txx}, \ldots$ by further x-differentiation. As for the second derivative u_{tt} at $t = 0$, we obtain it from the wave equation as $c^2 u_{xx} = c^2 (u_0)_{xx}$. Again, this also determines $u_{ttx}, u_{ttxx}, \ldots$ The third derivative u_{ttt} is simply $c^2 u_{txx} = c^2 (u_1)_{xx}$. For the fourth derivative u_{tttt}, apply the wave equation twice and get it as $c^4 (u_0)_{xxxx}$. And so on. Once the partial derivatives are known, so is u itself in a neighborhood of $t = 0$ by a Taylor expansion — this is the original argument behind the Cauchy-Kowalevsky theorem.

The same argument fails in characteristic coordinates. Indeed, assume that the equation is $u_{\xi\eta} + p u_\xi + q u_\eta + r u = 0$, and that the Cauchy data is $u(\xi, 0) = v_0(\xi)$, $u_\eta(\xi, 0) = v_1(\eta)$. Are the partial derivatives of u all determined in a unique manner at $\eta = 0$? We get u from v_0, as well as $u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \ldots$ by further ξ differentiation. We get u_η from v_1, as well as $u_{\eta\xi}, u_{\eta\xi\xi}, \ldots$ by further ξ differentiation. To make progress, we now need to consider the equation $u_{\xi\eta} + \text{(l.o.t.)} = 0$, but two problems arise:

- First, all the derivatives appearing in the equation have already been determined in terms of v_0 and v_1, and there is no reason to believe that this choice is compatible with the equation. In general, it isn't. There is a problem of existence.

- Second, there is no way to determine $u_{\eta\eta}$ from the equation, as this term does not appear. Hence additional data would be needed to determine this partial derivative. There is a problem of uniqueness.

The only way to redeem this existence-uniqueness argument is by making sure that the equation contains a $u_{\eta\eta}$ term, i.e., by making sure that η is *non*-characteristic.

Please refer to the exercise section for a link between characteristic equations, and the notions of traveltime and (light, sound) ray. We will return to such topics in the scope of geometrical optics, in chapter 6.

1.2.3 Spherical waves, Green's functions

Consider $x \in \mathbb{R}^3$ and c constant. We will only be dealing with solutions in 3 spatial dimensions for now. We seek radially symmetric solutions of the wave equation. In spherical coordinate (r, θ, ϕ), the Laplacian reads

$$\Delta u = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) + \text{ angular terms.}$$

For radially symmetric solutions of the wave equation, therefore,

$$\frac{\partial^2}{\partial t^2}(ru) = \frac{\partial^2}{\partial r^2}(ru).$$

This is a one-dimensional wave equation in the r variable, whose solution we derived earlier:

$$ru(r,t) = F(r-ct) + G(r+ct) \qquad \Rightarrow \qquad u(r,t) = \frac{F(r-ct)}{r} + \frac{G(r+ct)}{r}.$$

Spherical waves corresponding to the F term are called *outgoing*, while waves corresponding to the G term are called *incoming*. More generally, spherical waves can be outgoing/incoming with respect to any point $y \in \mathbb{R}^3$, for instance

$$u(x,t) = \frac{F(\|x-y\| - ct)}{\|x-y\|}.$$

Notice that we had already seen that $\|x-y\| \pm ct$ is a characteristic variable for the wave equation, in the previous section. The surfaces $\|x-y\| = ct+$ const. are often called *light cones* in the setting of electromagnetic waves.

In what follows we will be interested in the special case $F(r) = \delta(r)$, the Dirac delta, for which the wave equation is only satisfied in a distributional sense. Superpositions of such spherical waves are still solutions of the wave equation.

It turns out that *any* solution of the wave equation in $\mathbb{R}^3$, with constant c, can be written as a superposition of such spherical waves. Let us consider a quantity which is not quite the most general yet:

$$u(x,t) = \int_{\mathbb{R}^3} \frac{\delta(\|x-y\| - ct)}{\|x-y\|} \psi(y) dy. \tag{1.14}$$

Since $\|x-y\| = ct$ on the support of the delta function, the denominator can be written ct. Denoting by $B_x(ct)$ the ball centered at x and with radius ct, we can rewrite

$$u(x,t) = \frac{1}{ct} \int_{\partial B_x(ct)} \psi(y) dy.$$

hence the name spherical means (note that the argument of δ has derivative 1 in the radial variable — no Jacobian is needed.) The interesting question is that of matching $u(x,t)$ given by such a formula, with the initial conditions. By the mean value theorem,

$$u(x,t) \sim 4\pi ct\psi(x), \qquad t \to 0,$$

which tends to zero as $t \to 0$. On the other hand, an application of the Reynolds transport theorem (or a non-rigorous yet correct derivative in time of the equation above) yields

$$\lim_{t\to 0} \frac{\partial u}{\partial t}(x,t) = 4\pi c\psi(x).$$

We are therefore in presence of initial conditions $u_0 = 0$, and arbitrary $u_1 = 4\pi c\psi(x)$ arbitrary. In that case, *the* solution of the constant-c wave equation in $\mathbb{R}^3$ is

$$u(x,t) = \int G(x,y;t) u_1(y)\, dy,$$

with the so-called *Green's function*

$$G(x,y;t) = \frac{\delta(\|x-y\| - ct)}{4\pi c^2 t}, \qquad t > 0, \tag{1.15}$$

and zero when $t \leq 0$.

Let us now describe the general solution for the other situation when $u_1 = 0$, but $u_0 \neq 0$. The trick is to define $v(x,t)$ by the same formula (1.14), and consider $u(x,t) = \frac{\partial v}{\partial t}$, which also solves the wave equation:

$$\left[\frac{\partial^2}{\partial t^2} - c^2\Delta\right] \frac{\partial v}{\partial t} = \frac{\partial}{\partial t} \left[\frac{\partial^2}{\partial t^2} - c^2\Delta\right] v = 0.$$

The limits are now

$$\lim_{t\to 0} u(x,t) = 4\pi c\psi(x),$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial^2 v}{\partial t^2} = c^2 \Delta v, \qquad \lim_{t\to 0} c^2 \Delta v(x,t) = c^2 \Delta \lim_{t\to 0} v(x,t) = 0$$

(limit and derivative are interchangeable when the function is smooth enough.) The time derivative trick is all that is needed to generate the solution in the case $u_1 = 0$:

$$u(x,t) = \int \frac{\partial G}{\partial t}(x,y;t) u_0(y)\, dy.$$

The general solution is obtained by superposition of these two special cases:

$$u(x,t) = \int \left[\frac{\partial G}{\partial t}(x,y;t) u_0(y) + G(x,y;t) u_1(y) \right] dy. \tag{1.16}$$

The concept of Green' function G is much more general than suggested by the derivation above. Equation (1.16), for instance, holds in arbitrary dimension and for variable media, albeit with a different Green's function — a claim that we do not prove here. In two dimensions and constant c for instance, it can be shown[7] that

$$G(x,y;t) = \frac{1}{2\pi c\sqrt{c^2 t^2 - \|x-y\|^2}}, \qquad \text{when } t > 0,$$

and zero otherwise. In variable media, explicit formulas are usually not available.

In the wider context of linear PDE, Green's functions are more often introduced as linking a right-hand-side forcing f to the solution u upon integration. For a linear PDE $\mathcal{L}u = f$, Green's functions are to the differential operator $\mathcal{L}$ what the inverse matrix A^{-1} is to a matrix A. Accordingly, the Green's function describes the solution of the wave equation with a right-hand side forcing — a setting more often encountered in imaging than initial-value problems. The premise of the proposition below is that G is defined[8] through (1.16), even as $x \in \mathbb{R}^n$ and c is a function of x.

[7]By the so called "method of descent". See the book *Introduction to PDE* by Gerald Folland for a wonderful explanation of wave equations in constant media.

[8]The tables could be turned, and G could instead be defined by (1.17). In that case (1.16) would be a proposition.

Proposition 2. *(Duhamel principle) For $x \in \mathbb{R}^n$, and $t > 0$, the solution of the inhomogeneous problem*

$$\left[\frac{\partial^2}{\partial t^2} - c^2(x)\Delta\right] u(x,t) = f(x,t), \qquad u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0.$$

is

$$u(x,t) = \int_0^t \int G(x,y;t-s) f(y,s)\, dyds. \tag{1.17}$$

Proof. Let us check that the wave equation holds.

For each $s > 0$, consider the auxiliary problem

$$\left[\frac{\partial^2}{\partial t^2} - c^2(x)\Delta\right] v_s(x,t) = f(x,t), \qquad v_s(x,0) = 0, \qquad \frac{\partial v_s}{\partial t}(x,0) = f(x,s).$$

Then

$$v_s(x,t) = \int G(x,y;t) f(y,s)\, dy.$$

The candidate formula for u is

$$u(x,t) = \int_0^t v_s(x,t-s)\, ds.$$

Let us now check that this u solves the wave equation. For one, $u(x,0) = 0$ because the integral is over an interval of length zero. We compute

$$\frac{\partial u}{\partial t}(x,t) = v_s(x,t-s)|_{s=t} + \int_0^t \frac{\partial v_s}{\partial t}(x,t-s)\, ds = \int_0^t \frac{\partial v_s}{\partial t}(x,t-s)\, ds.$$

For the same reason as previously, $\frac{\partial u}{\partial t}(x,0) = 0$. Next,

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial v_s}{\partial t}(x,t-s)|_{s=t} + \int_0^t \frac{\partial^2 v_s}{\partial t^2}(x,t-s)\, ds \\
&= f(x,t) + \int_0^t c^2(x)\Delta v_s(x,t-s)\, ds \\
&= f(x,t) + c^2(x)\Delta \int_0^t v_s(x,t-s)\, ds \\
&= f(x,t) + c^2(x)\Delta u(x,t).
\end{aligned}$$

Since the solution of the wave equation is unique, the formula is general. □

Because the Green's function plays such a special role in the description of the solutions of the wave equation, it also goes by *fundamental solution*. We may specialize (1.17) to the case $f(x,t) = \delta(x-y)\delta(t)$ to obtain the equation that the Green's function itself satsfies,

$$\left[\frac{\partial^2}{\partial t^2} - c^2(x)\Delta_x\right] G(x,y;t) = \delta(x-y)\delta(t).$$

In the spatial-translation-invariant case, G is a function of $x-y$, and we may write $G(x,y;t) = g(x-y,t)$. In that case, the general solution of the wave equation with a right-hand side $f(x,t)$ is the space-time convolution of f with g.

A spatial dependence in the right-hand-side such as $\delta(x-y)$ may be a mathematical idealization, but the idea of a point disturbance is nevertheless a very handy one. In radar imaging for instance, antennas are commonly assumed to be point-like, whether on arrays or mounted on a plane/satellite. In exploration seismology, sources are often modeled as point disturbances as well (shots), both on land and for marine surveys.

The physical interpretation of the concentration of the Green's function along the cone $\|x-y\| = ct$ is called the *Huygens principle*. Starting from an initial condition at $t=0$ supported along (say) a curve Γ, this principle says that the solution of the wave equation is mostly supported on the envelope of the circles of radii ct centered at all the points on Γ.

1.2.4 The Helmholtz equation

It is often convenient to use a formulation of the wave equation in the frequency domain. If

$$\hat{u}(x,\omega) = \int e^{i\omega t} u(x,t)\, dt,$$

and if $\left[\frac{\partial^2}{\partial t^2} - c^2(x)\Delta_x\right] u = f$, then it is immediate to check that the (inhomogeneous) *Helmholtz equation* holds:

$$-\left[\omega^2 + c^2(x)\Delta\right] \hat{u}(x,\omega) = \hat{f}(x,\omega).$$

The notion of Green's function is also very useful for the Helmholtz equation: it is the function $\hat{G}(x,y;\omega)$ such that

$$\hat{u}(x,\omega) = \int \hat{G}(x,y;\omega)\hat{f}(y,\omega)\, dy.$$

It is a good exercise to check that $\hat{G}(x, y; \omega)$ is indeed the Fourier transform of $G(x, y; t)$ in t, by Fourier-transforming (1.17) and applying the convolution theorem. By specializing the Helmholtz equation to the right-hand side $\hat{f}(x, \omega) = \delta(x)$, we see that the Green's function itself obeys

$$-\left[\omega^2 + c^2(x)\Delta\right] \hat{G}(x, y; \omega) = \delta(x). \tag{1.18}$$

In particular, for $x \in \mathbb{R}^3$ and constant c, we get $(x \neq y)$

$$\begin{aligned}
\hat{G}(x, y; \omega) &= \int_0^\infty e^{i\omega t} \frac{\delta(\|x - y\| - ct)}{4\pi c^2 t} \, dt \\
&= \int_0^\infty e^{i\omega t} \frac{\delta(\|x - y\| - ct)}{4\pi c\|x - y\|} \, dt \\
&= \int_0^\infty e^{i\frac{\omega}{c}t'} \frac{\delta(\|x - y\| - t')}{4\pi\|x - y\|} \, dt' \\
&= \frac{e^{ik\|x-y\|}}{4\pi\|x - y\|}, \qquad k = \omega/c.
\end{aligned}$$

We will often use this form of the Green's function in the sequel. It is an outgoing spherical wave generated by a "point source" at $x = y$.

Note that $\omega \to -\omega$ corresponds to time reversal: $\frac{e^{-ik\|x-y\|}}{4\pi\|x-y\|}$ is also a solution of the Helmholtz equation for $x \neq y$, but it is an incoming rather than outgoing wave. The sign in the exponent depends on the choice of convention for the Fourier transform[9]

Some mathematical care should be exercised when posing the Helmholtz equation in free space. Uniqueness, in particular, is not as easy to guarantee as for the time-dependent wave equation. "Sufficient decay as $\|x\| \to \infty$" is not a good criterion for uniqueness, since we've just seen an example of two waves $\frac{e^{\pm i\omega\|x-y\|/c}}{4\pi\|x-y\|}$ which have the same modulus and obey the same equation (1.18). Instead, it is customary to require the wave to be *outgoing* in order to have a well-posed problem in constant c. We say that $\hat{u}(x, \omega)$ obeys the *Sommerfeld radiation condition* in $\mathbb{R}^3$ if $(r = \|x\|)$

$$\left(\frac{\partial}{\partial r} - ik\right) \hat{u}(x, \omega) = o(\frac{1}{|x|}),$$

[9] We choose $e^{i\omega t}$ for the direct transform, and $e^{-i\omega t}$ for the inverse transform, in accordance with practice in signal processing, radar imaging, and seismic imaging. For the spatial Fourier transforms, however, we adopt the opposite convention $e^{-ik\cdot x}$ for the direct transform, and $e^{ik\cdot x}$ for the inverse transform.

i.e., $\lim_{|x|\to\infty} |x| \left(\frac{\partial}{\partial r} - ik\right) \hat{u}(x,\omega) = 0$. It is a good exercise to check that $\hat{G}(x,y;\omega)$ obeys this radiation conditions, while $\hat{G}(x,y;-\omega)$ does not.

1.2.5 Reflected waves

Spatial variability in the physical parameters ($\rho, \kappa; \epsilon, \mu; \lambda, \mu$, etc.) entering the wave equation generate wave scattering, i.e., changes of the direction of propagation of the waves. Of particular interest are discontinuities, or other non-C^∞ singularities, which generate reflected waves alongside transmitted waves.

Let us study reflection and transmission in the 1D, variable-density acoustics equation

$$\frac{\partial^2 u}{\partial t^2} = \kappa(x) \frac{\partial}{\partial x} \left(\frac{1}{\rho(x)} \frac{\partial u}{\partial x} \right).$$

Consider a step discontinuity at $x = 0$, with $\rho(x) = \rho_1$ and $\kappa(x) = \kappa_1$ in $x < 0$, and $\rho(x) = \rho_2$ and $\kappa(x) = \kappa_2$ in $x > 0$. Assume an incident plane wave $u_i(x,t) = e^{i(k_1 x - \omega t)}$ in $x < 0$; we are interested in finding the reflection coefficient R and the transmission coefficient T so the solution reads

$$u_i(x,t) + u_r(x,t) = e^{i(k_1 x - \omega t)} + Re^{i(k_1 x + \omega t)}, \qquad x < 0.$$

$$u_t(x,t) = Te^{i(k_2 x - \omega t)}, \qquad x > 0.$$

The connection conditions are the continuity of u and $\frac{1}{\rho}\frac{\partial u}{\partial x}$. To justify this, remember that u is in fact a pressure disturbance in the acoustic case, while $\frac{1}{\rho}\frac{\partial u}{\partial x}$ is minus the time derivative of particle velocity, and these two quantities are continuous on physical grounds. There is also a mathematical justification for the continuity of $\frac{1}{\rho}\frac{\partial u}{\partial x}$: if it weren't, then $\frac{\partial}{\partial x}\left(\frac{1}{\rho(x)}\frac{\partial u}{\partial x}\right)$ would have a point mass (Dirac atom) at $x = 0$, which would pose a problem both for the multiplication by a discontinuous $\kappa(x)$, and because $\frac{\partial^2 u}{\partial t^2}$ is supposed to be a finite function, not a distribution.

At $x = 0$, the connection conditions give

$$1 + R = T,$$

$$\frac{1}{\rho_1}(-ik_1 - ik_1 R) = \frac{1}{\rho_2}(ik_2 T).$$

Eliminate k_1 and k_2 by expressing them as a function of ρ_1, ρ_2 only; for instance

$$\frac{k_1}{\rho_1} = \frac{\omega}{\rho_1 c_1} = \frac{\omega}{\sqrt{\rho_1 \kappa_1}},$$

and similarly for $\frac{k_2}{\rho_2}$. Note that ω is fixed throughout and does not depend on x. The quantity in the denominator is physically very important: it is $Z = \rho c = \sqrt{\kappa \rho}$, the *acoustic impedance*. The R and T coefficients can then be solved for as

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \qquad T = \frac{2Z_2}{Z_2 + Z_1}.$$

It is the *impedance jump* $Z_2 - Z_1$ which mostly determines the magnitude of the reflected wave.$R = 0$ corresponds to an impedance match, even in the case when the wave speeds differ in medium 1 and in medium 2.

The same analysis could have been carried out for a more general incoming wave $f(x - c_1 t)$, would have given rise to the same R and T coefficients, and to the complete solution

$$u(x,t) = f(x - c_1 t) + Rf(-x - c_1 t), \qquad x < 0, \tag{1.19}$$

$$u(x,t) = Tf(\frac{c_1}{c_2}(x - c_2 t)), \qquad x > 0. \tag{1.20}$$

The reader can check the relation

$$1 = R^2 + \frac{Z_1}{Z_2} T^2,$$

which corresponds to conservation of energy. An exercise in section 1.3 aims to establish this link. Note that $\mathcal{R} = R^2$ and $\mathcal{T} = \frac{Z_1}{Z_2} T^2$ are sometimes referred to as reflection and transmission coefficients, though they measure intensities rather than amplitudes. The intensity coefficients are even denoted as R and T in place of $\mathcal{R}$ and $\mathcal{T}$ in some texts.

Physically, the acoustic impedance Z is the proportionality constant between the pressure amplitude and the velocity amplitude of an acoustic wave. We do not have direct access to Z in the acoustic equations however, as $p(x,t) \neq Zv(x,t)$ pointwise – only combinations of partial derivatives match. So Z is in some sense an "averaged quantity" over at least a wavelength. On can derive the expression of Z from the time-harmonic regime. The first equation (1.1) in the acoustic system reads, in the (k, ω) domain (in one spatial dimension),

$$i\omega \hat{v}(k,\omega) = -\frac{1}{\rho_0} ik\hat{p}(k,\omega),$$

or, if we simplify further,

$$|\hat{p}| = Z|\hat{v}|, \qquad Z = \rho_0 c = \sqrt{\rho_0 \kappa_0}.$$

The same relation would have been obtained from (1.2). The larger Z, the more difficult to move particle from a pressure disturbance, i.e., the smaller the corresponding particle velocity.

The definition of acoustic impedance is intuitively in line with the traditional notion of electrical impedance for electrical circuits. To describe the latter, consider Ampère's law in the absence of a magnetic field:

$$\frac{\partial D}{\partial t} = -j \qquad \Rightarrow \qquad \epsilon \frac{\partial E}{\partial t} = -j.$$

In the time-harmonic setting (AC current), $i\omega\epsilon\hat{E} = -\hat{j}$. Consider a conducting material, for which the permittivity reduces to the conductivity:

$$\epsilon = i\frac{\sigma}{\omega}$$

It results that $\hat{E} = Z\hat{j}$ with the resistivity $Z = 1/\sigma$. This is the differential version of Ohm's law. The (differential) impedance is exactly the resistivity in the real case, and can accommodate capacitors and inductions in the complex case. Notice that the roles of E (or $V0$ and j (or I) in an electrical circuit are quite analogous to p and v in the acoustic case.

There are no waves in the conductive regime we just described, so it is out of the question to seek to write R and T coefficients, but reflections and transmissions of waves do occur at the interface between two dielectric materials. Such is the case of light propagating in a medium with variable index of reflection. To obtain the R and T coefficients in the optical case, the procedure is as follows:

- Consider Amp‘ere's law again, but this time with a magnetic field H (because it is needed to describe waves) but no current (because we are dealing with dielectrics):

  $$\frac{\partial D}{\partial t} = \nabla \times H.$$

  Use $D = \epsilon E$.

- Assume plane waves with complex exponentials, or in the form $E(k \cdot x - \omega t)$ and $H(k \cdot x - \omega t)$.

- Use continuity of $n \times E$ and $n \times H$ at the interface (tangential components).

- Assume no magnetism: $\mu =$ const.

The quantity of interest is not the impedance, but the index of refraction $n = \frac{1}{c} = \sqrt{\epsilon\mu}$. Further assuming that the waves are normally incident to the interface, we have

$$R = \frac{n_2 - n_1}{n_2 + n_1}, \qquad T = \frac{2n_2}{n_2 + n_1}.$$

These relations become more complicated when the angle of incidence is not zero. In that case R and T also depend on the polarization of the light. The corresponding equations for R and T are then called Fresnel's equations. Their expression and derivation can be found in "Principles of optics" by Born and Wolf.

1.3 Exercises

1. Continue the reasoning in section 1.1.1 with the entropy to justify the equations of variable-density acoustics. [Hints: conservation of entropy reads $\frac{\partial s}{\partial t} + v \cdot \nabla s = 0$. Continue assuming that the background velocity field is $v_0 = 0$. Assume a fixed, variable background density $\rho_0(x)$. The new constitutive relation is $p = f(\rho, s)$. Consider defining $c^2(x) = \frac{\partial f}{\partial \rho}(\rho_0(x), s_0(x))$.]

2. First, show the multivariable rule of integration by parts $\int \nabla f \cdot g = -\int f \, \nabla \cdot g$, when f and g are smooth and decay fast at infinity, by invoking the divergence theorem. Second, use this result to show that $L^* = -L$ for variable-density acoustics (section 1.1.1), i.e., show that $\langle Lw, w' \rangle = -\langle w, Lw' \rangle$ for all reasonable functions w and w', and where $\langle \cdot, \cdot \rangle$ is the adequate notion of inner product seen in section 1.1.1.

3. Show that $\langle Lw, w' \rangle = -\langle w, Lw' \rangle$ for general elastic waves.

4. In $\mathbb{R}^2$, consider

$$f_\omega(x) = \int_0^{2\pi} e^{ik_\theta \cdot x} d\theta, \qquad k_\theta = |k| \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$

with $|k| = \omega/c$. Show that f_ω is a solution of the homogeneous Helmholtz equation (1.10) with constant c, and simplify the expression of f_ω by means of a Bessel function. [Hint: show first that f_ω is radially symmetric.]

5. Find all the functions $\tau(x)$ for which

$$\xi(x,t) = \tau(x) - t$$

is a solution of (1.12) in the case $x \in \mathbb{R}$.

The function $\tau(x)$ has the interpretation of a *traveltime*.

6. Consider a characteristic curve as the level set $\xi(x,t)$ =const., where ξ is a characteristic coordinate obeying (1.12). Express this curve parametrically as $(X(t), t)$, and find a differential equation for $X(t)$ of the form $\dot{X}(t) = \ldots$ How do you relate this $X(t)$ to the traveltime function $\tau(x)$ of the previous exercise? Justify your answer.

Such functions $X(t)$ are exactly the *rays* — light rays or sound rays. They encode the idea that waves propagate with local speed $c(x)$.

7. Give a complete solution to the wave equation in $\mathbb{R}^n$,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u, \qquad u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x),$$

by Fourier-transforming $u(x,t)$ in the x-variable, solving the resulting ODE to obtain the $e^{\pm i|k|/ct}$ time dependencies, matching the initial conditions, and finishing with an inverse Fourier transform. The resulting formula is a generalization of d'Alembert's formula.

8. We have seen the expression of the wave equation's Green function in the (x,t) and (x,ω) domains. Find the expression of the wave equation's Green function in the (ξ,t) and (ξ,ω) domains, where ξ is dual to x. [Hint: the previous exercise involves the Green's function in the (ξ,t) domain. The Green's function in (ξ,ω) is called propagator (in momentum space) by quantum physicists.]

9. Check that the relation $1 = R^2 + \frac{Z_1}{Z_2} T^2$ for the reflection and transmission coefficiets follows from conservation of energy for acoustic waves. [Hint: use the definition of energy given in section 1.1.1, and the general form (1.19, 1.20) of a wavefield scattering at a jump interface in one spatial dimension.]

10. The wave equation (2.2) can be written as a first-order system

$$M \frac{\partial w}{\partial t} - Lw = \widetilde{f},$$

with

$$w = \begin{pmatrix} \partial u / \partial t \\ \nabla u \end{pmatrix}, \qquad M = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & \nabla \cdot \\ \nabla & 0 \end{pmatrix}, \qquad \widetilde{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

First, check that $L^* = -L$ for the L^2 inner product $\langle w, w' \rangle = \int (w_1 w_1' + w_2 w_2') \, dx$ where $w = (w_1, w_2)^T$. Second, check that $E = \langle w, Mw \rangle$ is a conserved quantity.

11. Another way to write the wave equation (2.2) as a first-order system is

$$M \frac{\partial w}{\partial t} - Lw = \widetilde{f},$$

with

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad M = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \qquad \widetilde{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

First, check that $L^* = -L$ for the inner product $\langle w, w' \rangle = \int (\nabla u \cdot \nabla u' + v v') \, dx$. Second, check that $E = \langle w, Mw \rangle$ is a conserved quantity.

Chapter 2

Scattering series

In this chapter we describe the nonlinearity of the map $c \mapsto u$ in terms of a perturbation (Taylor) series. To first order, the linearization of this map is called the Born approximation. Linearization and scattering series are the basis of most inversion methods, both direct and iterative.

The idea of perturbation permeates imaging for physical reasons as well. In radar imaging for instance, the background velocity is $c_0 = 1$ (speed of light), and the *reflectivity* of scatterers is viewed as a deviation in $c(x)$. The assumption that $c(x)$ does not depend on t is a strong one in radar: it means that the scatterers do not move. In seismology, it is common to consider a smooth background velocity $c_0(x)$ (rarely well known), and explain the scattered waves as reflections due to a "rough" (singular/oscillatory) perturbations to this background. In both cases, we will write

$$\frac{1}{c^2(x)} = m(x), \qquad \frac{1}{c_0^2(x)} = m_0(x), \qquad m \text{ for "model"},$$

and, for some small number ϵ,

$$m(x) = m_0(x) + \epsilon m_1(x). \tag{2.1}$$

Note that, when perturbing $c(x)$ instead of $m(x)$, an additional Taylor approximation is necessary:

$$c(x) = c_0(x) + \epsilon c_1(x) \qquad \Rightarrow \qquad \frac{1}{c^2(x)} \simeq \frac{1}{c_0^2(x)} - 2\epsilon \frac{c_1(x)}{c_0^3(x)}.$$

While the above is common in seismology, we avoid making unnecessary assumptions by choosing to perturb $m(x) = 1/c^2(x)$ instead.

Perturbations are of course not limited to the wave equation with a single parameter c. The developments in this chapter clearly extend to more general wave equations.

2.1 Perturbations and Born series

Let

$$m(x)\frac{\partial^2 u}{\partial t^2} - \Delta u = f(x,t), \tag{2.2}$$

with zero initial conditions and $x \in \mathbb{R}^n$. Perturb $m(x)$ as in (2.1). The wavefield u correspondingly splits into

$$u(x) = u_0(x) + u_{sc}(x),$$

where u_0 solves the wave equation in the undisturbed medium m_0,

$$m_0(x)\frac{\partial^2 u_0}{\partial t^2} - \Delta u_0 = f(x,t). \tag{2.3}$$

We say u is the total field, u_0 is the incident field[1], and u_{sc} is the scattered field, i.e., anything but the incident field.

We get the equation for u_{sc} by subtracting (2.3) from (2.2), and using (2.1):

$$m_0(x)\frac{\partial^2 u_{sc}}{\partial t^2} - \Delta u_{sc} = -\epsilon\, m_1(x)\frac{\partial^2 u}{\partial t^2}. \tag{2.4}$$

This equation is implicit in the sense that the right-hand side still depends on u_{sc} through u. We can nevertheless reformulate it as an implicit integral relation by means of the Green's function:

$$u_{sc}(x,t) = -\epsilon \int_0^t \int_{\mathbb{R}^n} G(x,y;t-s) m_1(y) \frac{\partial^2 u}{\partial t^2}(y,s)\, dyds.$$

Abuse notations slightly, but improve conciseness greatly, by letting

- G for the operator of space-time integration against the Green's function, and

[1]Here and in the sequel, u_0 is not the initial condition. It is so prevalent to introduce the source as a right-hand side f in imaging that it is advantageous to free the notation u_0 and reserve it for the incident wave.

- m_1 for the operator of multiplication by m_1.

Then $u_{sc} = -\epsilon \, G \, m_1 \, \frac{\partial^2 u}{\partial t^2}$. In terms of u, we have the implicit relation

$$u = u_0 - \epsilon \, G \, m_1 \, \frac{\partial^2 u}{\partial t^2},$$

called a *Lippmann-Schwinger* equation. The field u can be formally[2] expressed in terms of u_0 by writing

$$u = \left[I + \epsilon \, G \, m_1 \, \frac{\partial^2}{\partial t^2} \right]^{-1} u_0. \tag{2.5}$$

While this equation is equivalent to the original PDE, it shines a different light on the underlying physics. It makes explicit the link between u_0 and u, as if u_0 "generated" u via scattering through the medium perturbation m_1.

Writing $[I + A]^{-1}$ for some operator A invites a solution in the form of a Neumann series $I - A + A^2 - A^3 + \ldots$, provided $\|A\| < 1$ in some norm. In our case, we write

$$u = u_0 - \epsilon \left(G \, m_1 \, \frac{\partial^2}{\partial t^2} \right) u_0 + \epsilon^2 \left(G \, m_1 \, \frac{\partial^2}{\partial t^2} \right) \left(G \, m_1 \, \frac{\partial^2}{\partial t^2} \right) u_0 + \ldots$$

This is called a *Born series*. The proof of convergence, based on the "weak scattering" condition $\epsilon \| G \, m_1 \, \frac{\partial^2}{\partial t^2} \|_* < 1$, in some norm to be determined, will be covered in the next section. It retroactively justifies why one can write (2.5) in the first place.

The Born series carries the physics of *multiple scattering*. Explicitly,

$$\begin{aligned}
u = {} & u_0 \qquad \text{(incident wave)} \\
& - \epsilon \int_0^t \int_{\mathbb{R}^n} G(x, y; t - s) m_1(y) \frac{\partial^2 u_0}{\partial t^2}(y, s) \, dy ds \\
& \qquad \text{(single scattering)} \\
& + \epsilon^2 \int_0^t \int_{\mathbb{R}^n} G(x, y_2; t - s_2) m_1(y_2) \frac{\partial^2}{\partial s_2^2} \left[\int_0^{s_2} \int_{\mathbb{R}^n} G(y_2, y_1; s_2 - s_1) m_1(y_1) \frac{\partial^2 u_0}{\partial t^2}(y_1, s_1) \, dy_1 ds_1 \right] dy_2 ds_2 \\
& \qquad \text{(double scattering)} \\
& + \ldots
\end{aligned}$$

[2]For mathematicians, "formally" means that we are a step ahead of the rigorous exposition: we are only interested in inspecting the *form* of the result before we go about proving it. That's the intended meaning here. For non-mathematicians, "formally" often means rigorous, i.e., the opposite of "informally"!

We will naturally summarize this expansion as

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots$$

where ϵu_1 represent single scattering, $\epsilon^2 u_2$ double scattering, etc. For instance, the expression of u_1 can be physically read as "the incident wave initiates from the source at time $t = 0$, propagates to y where it scatters due to $m(y)$ at time $t = s$, then further propagates to reach x at time t." The expression of u_2 can be read as "the incident wave initiates from the source at $t = 0$, propagates to y_1 where it first scatters at time $t = s_1$, them propagates to y_2 where it scatters a second time at time $t = s_2$, then propagates to x at time t, where it is observed." Since scatterings are not a priori prescribed to occur at fixed points in space and time, integrals must be taken to account for all physically acceptable scattering scenarios.

The approximation

$$u_{sc}(x) \simeq \epsilon u_1(x)$$

is called the *Born approximation*. From $u_1 = -G m_1 \frac{\partial^2 u_0}{\partial t^2}$, we can return to the PDE and obtain the equation for the primary reflections:

$$m_0(x) \frac{\partial^2 u_1}{\partial t^2} - \Delta u_1 = -m_1(x) \frac{\partial^2 u_0}{\partial t^2}. \qquad (2.6)$$

The only difference with (2.4) is the presence of u_0 in place of u in the right-hand side (and ϵ is gone, by choice of normalization of u_1). Unlike (2.4), equation (2.6) is explicit: it maps m_1 to u_1 in a linear way. The incident field u_0 is determined from m_0 alone, hence "fixed" for the purpose of determining the scattered fields.

The Born series can be seen as a Taylor series of the *forward map*

$$u = \mathcal{F}[m],$$

in the sense of the calculus of variations. Denote by $\frac{\delta \mathcal{F}}{\delta m}[m_0]$ the functional derivative of $\mathcal{F}$ with respect to m, evaluated at m_0 — an operator acting from model space (m) to data space (u). Denote by $\frac{\delta^2 \mathcal{F}}{\delta m^2}[m_0]$ the functional Hessian — a bilinear form from model space to data space (see the appendix for background on functional derivatives). Then we claim that

$$u = u_0 + \epsilon \frac{\delta \mathcal{F}}{\delta m}[m_0]\, m_1 + \frac{\epsilon^2}{2} \langle \frac{\delta^2 \mathcal{F}}{\delta m^2}[m_0]\, m_1, m_1 \rangle + \ldots$$

It is convenient to write the *linearized forward map* as

$$F = \frac{\delta \mathcal{F}}{\delta m}[m_0],$$

or also as $\frac{\partial u}{\partial m}$. The point of F is that it makes explicit the linear link between m_1 and u_1:

$$u_1 = F m_1.$$

While $\mathcal{F}$ is supposed to completely model data (up to measurement errors), F would properly explain data only in the regime of the Born approximation.

Let us show that the two concepts of linearized scattered field coincide, namely

$$u_1 = \frac{\delta \mathcal{F}}{\delta m}[m_0]\, m_1 = G m_1 \frac{\partial u_0}{\partial t^2}.$$

This will justify the first term in the Taylor expansion above. For this purpose, take the $\frac{\delta}{\delta m}$ derivative of (2.2), evaluate it at m_0, and check that (2.6) holds for the resulting field. As previously, write $u = \mathcal{F}(m)$ and $F = \frac{\delta \mathcal{F}}{\delta m}[m]$. We get the operator-value equation

$$\frac{\partial^2 u}{\partial t^2} I + m \frac{\partial^2}{\partial t^2} F - \Delta F = 0.$$

Evaluate the functional derivatives at the base point m_0, so that $u = u_0$. Applying each term as an operator to the function m_1, and defining $u_1 = F m_1$, we obtain

$$m_1 \frac{\partial^2 u_0}{\partial t^2} + \frac{\partial^2 u_1}{\partial t^2} u_1 - \Delta u_1 = 0,$$

which is exactly (2.6).

2.2 Convergence and accuracy of linearization

There are two answers to the question of justifying convergence of the Born series: a mathematical one and a physical one. The latter is definitely needed: our current mathematical understanding (2012) is not yet at a point where it can completely justify the physical intuition.

Let us describe what is known mathematically about convergence of Born series in a simple setting. To keep the notations concise, it is more convenient to treat the wave equation in first-order hyperbolic form

$$M\frac{\partial w}{\partial t} - Lw = f, \qquad L^* = -L,$$

for some inner product $\langle w, w' \rangle$. The conserved energy is then $E = \langle w, Mw \rangle$. See one of the exercises at the end of chapter 1 to illustrate how the wave equation can be put in precisely this form, with $\langle w, w' \rangle$ the usual L^2 inner product.

Consider a background medium M_0, so that $M = M_0 + \epsilon M_1$. Let $w = w_0 + \epsilon w_1 + \ldots$ Calculations very similar to those of the previous section (a good exercise) show that

- The Lippmann-Schwinger equation is

  $$w = w_0 - \epsilon G M_1 \frac{\partial w}{\partial t},$$

  with the Green's function $G = (M_0 \frac{\partial}{\partial t} - L)^{-1}$.

- The Neumann series of interest is

  $$w = w_0 - \epsilon G M_1 \frac{\partial w_0}{\partial t} + \epsilon^2 G M_1 \frac{\partial}{\partial t} G M_1 \frac{\partial w_0}{\partial t} + \ldots$$

  We identify $w_1 = -GM_1 \frac{\partial w_0}{\partial t}$.

- In differential form, the equations for the incident field w_0 and the primary scattered field w_1 are

  $$M_0 \frac{\partial w_0}{\partial t} - Lw_0 = f, \qquad M_0 \frac{\partial w_1}{\partial t} - Lw_1 = -M_1 \frac{\partial w_0}{\partial t},$$

- Convergence of the Born series occurs when

  $$\epsilon \| G M_1 \frac{\partial}{\partial t} \|_* < 1,$$

  in some induced operator norm, i.e., when $\epsilon \|w_1\|_* < \|w_0\|_*$ for arbitrary w_0, and $w_1 = -GM_1 \frac{\partial w_0}{\partial t}$, for some norm $\| \cdot \|_*$.

Notice that the condition $\epsilon\|w_1\|_* < \|w_0\|_*$ is precisely one of weak scattering, i.e., that the primary reflected wave ϵw_1 is weaker than the incident wave w_0.

While any induced norm over space and time in principle works for the proof of convergence of the Neumann series, it is convenient to use

$$\|w\|_* = \max_{0\leq t\leq T} \sqrt{\langle w, M_0 w\rangle} = \max_{0\leq t\leq T} \|\sqrt{M_0}w\|_2.$$

We compute

$$\begin{aligned}
\frac{d}{dt}\langle w_1, M_0 w_1\rangle &= 2\langle w_1, M_0 \frac{\partial w_1}{\partial t}\rangle \\
&= 2\langle w_1, L w_1 - M_1 \frac{\partial w_0}{\partial t}\rangle \\
&= -2\langle w_1, M_1 \frac{\partial w_0}{\partial t}\rangle \qquad \text{because } L^* = -L \\
&= -2\langle \sqrt{M_0} w_1, \frac{M_1}{\sqrt{M_0}} \frac{\partial w_0}{\partial t}\rangle.
\end{aligned}$$

The left-hand-side is also $\frac{d}{dt}\langle w_1, M_0 w_1\rangle = 2\|\sqrt{M_0}w_1\|_2 \frac{d}{dt}\|\sqrt{M_0}w_1\|_2$. By Cauchy-Schwarz, the right-hand-side is majorized by

$$2\|\sqrt{M_0}w_1\|_2 \, \|\frac{M_1}{\sqrt{M_0}}\frac{\partial w_0}{\partial t}\|_2.$$

Hence

$$\frac{d}{dt}\|\sqrt{M_0}w_1\|_2 \leq 2\|\frac{M_1}{\sqrt{M_0}}\frac{\partial w_0}{\partial t}\|_2.$$

$$\|\sqrt{M_0}w_1\|_2 \leq 2\int_0^t \|\frac{M_1}{\sqrt{M_0}}\frac{\partial w_0}{\partial t}\|_2(s)\, ds.$$

$$\begin{aligned}
\|w_1\|_* = \max_{0\leq t\leq T}\|\sqrt{M_0}w_1\|_2 &\leq 2T \max_{0\leq t\leq T}\|\frac{M_1}{\sqrt{M_0}}\frac{\partial w_0}{\partial t}\|_2 \\
&\leq 2T\|\frac{M_1}{M_0}\|_\infty \max_{0\leq t\leq T}\|\sqrt{M_0}\frac{\partial w_0}{\partial t}\|_2.
\end{aligned}$$

This last inequality is almost, but not quite, what we need. The right-hand side involves $\frac{\partial w_0}{\partial t}$ instead of w_0. Because time derivatives can grow arbitrarily large in the high-frequency regime, there is no hope of closing the argument without an additional assumption. We therefore require that w_0 be *bandlimited*, i.e., its Fourier transform in time should be compactly supported in the interval $[-\Omega, \Omega]$. In that case, we can invoke a classical result known as Bernstein's inequality, which says that $\|f'\|_\infty \leq \Omega \|f\|_\infty$ for all Ω-bandlimited f. (The same inequality holds with the L^p norm for all $1 \leq p \leq \infty$.) Then

$$\|w_1\|_* \leq 2\Omega T \|\frac{M_1}{M_0}\|_\infty \|w_0\|_*.$$

In view of our request that $\epsilon\|w_1\|_* < \|w_0\|_*$, it suffices to require

$$2\epsilon\,\Omega T\,\|\frac{M_1}{M_0}\|_\infty < 1.$$

This is only achieved if ϵ is small. A large frequency cutoff Ω and a large total propagation time T tend to make this "weak scattering" condition harder to satisfy. This situation is generic.

Note that the beginning of the argument, up to the Cauchy-Scwharz inequality, is called an *energy estimate* in math. It is a prevalent method to control the size of the solution many initial-value PDE.

See the book by Colton and Kress for a different analysis that takes into account the diameter of the support of M_1.

2.3 Exercises

1. Repeat the development of section (2.1) in the frequency domain (ω) rather than in time.

2. Derive Born series with a multiscale expansion: write $m = m_0 + \epsilon m_1$, $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots$, substitute in the wave equation, and equate like powers of ϵ.

3. Differentiate (2.2) twice with respect to m to obtain the functional Hessian of u with respect to m. As a result, find an expression for u_2 as the solution of a new linearized wave equation with u_1 in the right-hand side. This expression of the "wave-equation Hessian" is important later on as we describe accelerated descent methods for the inversion problem.

4. Write the Born series for the acoustic system, i.e., find the linearized equations that the first few terms obey.

Chapter 3

Adjoint-state methods

Chapter 4

Synthetic-aperture radar

Chapter 5

Computerized tomography

Chapter 6

Seismic imaging

Chapter 7

Optimization

Appendix A

Calculus of variations, functional derivatives

The calculus of variations is to multivariable calculus what functions are to vectors. It answers the question of how to differentiate with respect to functions, i.e., objects with an uncountable, infinite number of degrees of freedom.

Let X be some function space, endowed with a norm (technically, a Banach space). A *functional* ϕ is a map from X to $\mathbb{R}$. We denote its action on a function f as $\phi(f)$. An *operator* F is a map from X to X. We denote its action on a function f as Ff.

We say that a functional ϕ is Fréchet differentiable at $f \in X$ when there exists a linear functional A such that

$$\lim_{h \to 0} \frac{|\phi(f+h) - \phi(f) - A(h)|}{\|h\|} = 0.$$

If this relation holds, we say that A is the *functional derivative*, or Fréchet derivative, of ϕ at f, and we denote it as

$$A = \frac{\delta \phi}{\delta f}[f].$$

It is also called the *first variation* of ϕ. It is the equivalent of the gradient in multivariable calculus. The fact that A is a map from X to $\mathbb{R}$ corresponds to the idea that a gradient maps vectors to scalar when paired with the dot product; e.g. if $X = \mathbb{R}^n$ and $f = (f_1, \ldots, f_n)$, we have

$$\frac{\delta \phi}{\delta f}[f](h) = \nabla \phi(f) \cdot h.$$

For this reason, it is is also fine to write $A(h) = \langle A, h \rangle$.

The differential ratio formula for $\frac{\delta \phi}{\delta f}$ is called Gâteaux derivative,

$$\frac{\delta \phi}{\delta f}[f](h) = \lim_{t \to 0} \frac{\phi(f + th) - \phi(f)}{t}, \tag{A.1}$$

which corresponds to the idea of the directional derivative in $\mathbb{R}^n$.

Examples of functional derivatives:

- $\phi(f) = \langle g, f \rangle$,

  $$\frac{\delta \phi}{\delta f}[f] = g, \qquad \frac{\delta \phi}{\delta f}[f](h) = \langle g, h \rangle$$

  Because ϕ is linear, $\frac{\delta \phi}{\delta f} = \phi$. Proof: $\phi(f + th) - \phi(f) = \langle g, f + th \rangle - \langle g, f \rangle = t \langle g, h \rangle$, then use (A.1).

- $\phi(f) = f(x_0)$,

  $$\frac{\delta \phi}{\delta f}[f] = \delta(x - x_0), \qquad \text{(Dirac delta)}.$$

  This is the special case when $g(x) = \delta(x - x_0)$. Again, $\frac{\delta \phi}{\delta f} = \phi$.

- $\phi(f) = \langle g, f^2 \rangle$,

  $$\frac{\delta \phi}{\delta f}[f] = 2fg.$$

  Proof: $\phi(f + th) - \phi(f) = \langle g, (f + th)^2 \rangle - \langle g, f \rangle = t \langle g, 2fh \rangle + O(t^2) = t \langle 2fg, h \rangle + O(t^2)$, then use (A.1).

Nonlinear operators can also be differentiated with respect to their input function. We say $\mathcal{F} : X \to X$ is Fréchet differentiable when there exists a linear operator $F : X \to X$

$$\lim_{h \to 0} \frac{\| \mathcal{F}(f + h) - \mathcal{F}(f) - Fh \|}{\| h \|} = 0.$$

F is the functional derivative of $\mathcal{F}$, and we write

$$F = \frac{\delta \mathcal{F}}{\delta f}[f].$$

We still have the difference formula

$$\frac{\delta \mathcal{F}}{\delta f}[f]h = \lim_{t \to 0} \frac{\mathcal{F}(f + th) - \mathcal{F}(f)}{t}.$$

Examples:

- $\mathcal{F}(f) = f$. Then
$$\frac{\delta \mathcal{F}}{\delta f}[f] = I,$$
the identity. Proof: $\mathcal{F}$ is linear hence equals its functional derivative. Alternatively, apply the difference formula to get $\frac{\delta \mathcal{F}}{\delta f}[f]h = h$.

- $\mathcal{F}(f) = f^2$. Then
$$\frac{\delta \mathcal{F}}{\delta f}[f] = 2f,$$
the operator of multiplication by $2f$.

Under a suitable smoothness assumption, the Fréchet Hessian of an operator F can also be defined: it takes two functions as input, and returns a function in a linear manner ("bilinear operator"). It is defined through a similar finite-difference formula

$$\langle \frac{\delta^2 \mathcal{F}}{\delta f^2}[f]h_1, h_2 \rangle = \lim_{t \to 0} \frac{\mathcal{F}(f + t(h_2 + h_1)) - \mathcal{F}(f + th_2) - \mathcal{F}(f + th_1) + \mathcal{F}(f)}{t^2}.$$

The Hessian is also called second variation of $\mathcal{F}$.

Functional derivatives obey all the properties of multivariable calculus, such as chain rule and derivative of a product (when all the parties are sufficiently differentiable). A handy way to deal with functional derivatives in practice is to view them as gradients, and keep track of free vs. summation variables.

Functional derivatives are used to formulate linearized forward models for imaging, as well as higher-order terms in Born series. They are also useful for finding stationary-point conditions of Lagrangians, and gradient descent directions in optimization.