18.085 :: Linear algebra cheat sheet :: Spring 2014

1. Rectangular m-by-n matrices
   - Rule for transposes:
     $$(AB)^T = B^T A^T$$
   - The matrix $A^T A$ is always square, symmetric, and positive semi-definite. If in addition A has linearly independent columns, then $A^T A$ is positive definite.
   - Take C a diagonal matrix with positive elements. Then $A^T CA$ is always square, symmetric, and positive semi-definite. If in addition A has linearly independent columns, then $A^T CA$ is positive definite.
2. Square n-by-n matrices
   - Eigenvalues and eigenvectors are defined only for square matrices:
     $$Av = \lambda v$$
     The eigenvalues may be complex. There may be fewer than n eigenvectors (in that case we say the matrix is defective).
   - In matrix form, we can always write
     $$AS = S\Lambda,$$
     where S has the eigenvectors in the columns (S could be a rectangular matrix), and Λ is diagonal with the eigenvalues on the diagonal.
   - If A has n eigenvectors (a full set), then S is square and invertible, and
     $$A = S\Lambda S^{-1}.$$
   - If eigenvalues are counted with their multiplicity, then
     $$\text{tr}(A) = A_{11} + \ldots + A_{nn} = \lambda_1 + \ldots + \lambda_n.$$
   - If eigenvalues are counted with their multiplicity, then
     $$\det(A) = \lambda_1 \times \ldots \times \lambda_n.$$
   - Addition of a multiple of the identity shifts the eigenvalues:
     $$\lambda_j(A + cI) = \lambda_j(A) + c.$$
     No such rule exists in general when adding $A + B$.
   - Gaussian elimination gives
     $$A = LU.$$
     L is lower triangular with ones on the diagonal, and with the elimination multipliers below the diagonal. U is upper triangular with the pivots on the diagonal.
   - A is invertible if either of the following criteria is satisfied: there are n nonzero pivots; the determinant is not zero; the eigenvalues are all nonzero; the columns are linearly independent; the only way to have $Au = 0$ is if $u = 0$.
   - Conversely, if either of these criteria is violated, then the matrix is singular (not invertible).

3. Invertible matrices.

   - The system $Au = f$ has a unique solution $u = A^{-1}f$.
   - $(AB)^{-1} = B^{-1}A^{-1}$.
   - $(A^T)^{-1} = (A^{-1})^T$.
   - If $A = S\Lambda S^{-1}$, then $A^{-1} = S\Lambda^{-1}S^{-1}$ (same eigenvectors, inverse eigenvalues).

4. Symmetric matrices ($A^T = A$.)

   - Eigenvalues are real, eigenvectors are orthogonal, and there are always n eigenvectors (full set). (Precision concerning eigenvectors: accidentally, they may not come as orthogonal. But that case always corresponds to a multiple eigenvalue. There exists another choice of eigenvectors such that they are orthogonal.)
   - Can write
   $$A = Q\Lambda Q^T,$$
   where $Q^{-1} = Q^T$ (orthonormal matrix).
   - Can modify $A = LU$ into $A = LDL^T$ with diagonal D, and the pivots on the diagonal of D.
   - Can define positive definite matrices only in the symmetric case. A symmetric matrix is positive definite if either of the following criteria holds: all the pivots are positive; all the eigenvalues are positive; all the upper-left determinants are positive; or $x^TAx > 0$ unless $x = 0$.
   - Similar definition for positive semi-definite. The creiteria are: all the pivots are nonnegative; all the eigenvalues are nonnegative; all the upper-left determinants are nonnegative; or $x^TAx \geq 0$.
   - If A is positive definite, we have the Cholesky decomposition
   $$A = R^TR,$$
   which corresponds to the choice $R = \sqrt{D}L^T$.

5. Skew-symmetric matrices ($A^T = -A$.)

   - Eigenvalues are purely imaginary, eigenvectors are orthogonal, and there are always n eigenvectors (full set).