Matrix-vector multiplication using the FFT

Alex Townsend

There are a few special $n \times n$ matrices that can be applied to a vector in $\mathcal{O}(n \log n)$ operations.

1 Circulant

An $n \times n$ circulant matrix takes the form:

$$C = \begin{pmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{pmatrix}.$$

This matrix has the wonderful property of being diagonalized by the DFT matrix. That is,

$$C = F^{-1} \Lambda F,$$

where F is the $n \times n$ DFT matrix and Λ is a diagonal matrix such that $\Lambda = \mathrm{diag}(F\underline{c})$. Therefore a circulant matrix can be applied to a vector in $\mathcal{O}(n \log n)$ operations using the FFT.

2 Toeplitz

An $n \times n$ Toeplitz matrix takes the form:

$$T = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{pmatrix}$$

A toeplitz matrix can be embedded into a circulant matrix of size $2n$:

$$A = \left(\begin{array}{ccccc|ccccc}
a_0 & a_{-1} & \cdots & \cdots & a_{-n+1} & 0 & a_{n-1} & \cdots & a_2 & a_1 \\
a_1 & a_0 & \ddots & & \vdots & a_{-n+1} & \ddots & \ddots & & a_2 \\
a_2 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & & \ddots & \vdots \\
\vdots & & \ddots & a_0 & a_{-1} & a_{-2} & & \ddots & \ddots & a_{n-1} \\
a_{n-1} & \cdots & \cdots & a_1 & a_0 & a_{-1} & a_{-2} & \cdots & a_{-n+1} & 0 \\
\hline
0 & a_{n-1} & \cdots & \cdots & a_1 & a_0 & a_{-1} & \cdots & \cdots & a_{-n+1} \\
a_{-n+1} & \ddots & \ddots & & a_2 & a_1 & a_0 & \ddots & & \vdots \\
\vdots & \ddots & & \ddots & \vdots & a_2 & \ddots & \ddots & \ddots & \vdots \\
a_{-2} & & \ddots & \ddots & a_{n-1} & \vdots & & \ddots & a_0 & a_{-1} \\
a_{-1} & a_{-2} & \cdots & \cdots & 0 & a_{n-1} & \cdots & \cdots & a_1 & a_0
\end{array}\right).$$

To achieve a fast matrix-vector product we can use the relation:

$$T\underline{v} = \begin{pmatrix} I_n & 0_n \end{pmatrix} A \begin{pmatrix} \underline{v} \\ 0_n \end{pmatrix},$$

where the matrix-product with A can be performed using the FFT.

3 Hankel

An $n \times n$ Hankel matrix takes the form:

$$H = \begin{pmatrix}
b_0 & b_1 & b_2 & \cdots & \cdots & b_{n-1} \\
b_1 & b_2 & \iddots & \iddots & \iddots & b_{-1} \\
b_2 & \iddots & \iddots & \iddots & \iddots & b_{-2} \\
\vdots & \iddots & \iddots & & & \vdots \\
\vdots & \iddots & \iddots & & b_{-n+3} & b_{-n+2} \\
b_{n-1} & b_{-1} & \cdots & \cdots & b_{-n+2} & b_{-n+1}
\end{pmatrix}.$$

If the columns are permuted left-to-right, then H becomes a Toeplitz matrix. Therefore, we can use the following relation:

$$H\underline{v} = T\underline{v}(n\!:\!-1\!:\!1),$$

where T is a Toeplitz matrix.