On Chebyshev interpolation of analytic functions

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Abstract

This paper reviews the notion of interpolation of a smooth function by means of Chebyshev polynomials, and the well-known associated results of spectral accuracy when the function is analytic. The rate of decay of the error is proportional to ρ^{-N}, where ρ is a bound on the elliptical radius of the ellipse in which the function has a holomorphic extension. An additional theorem is provided to cover the situation when only bounds on the derivatives of the function are known.

1 Review of Chebyshev interpolation

The Chebyshev interpolant of a function f on $[-1, 1]$ is a superposition of Chebyshev polynomials $T_n(x)$,

$$p(x) = \sum_{n=0}^{N} c_n T_n(x),$$

which interpolates f in the sense that $p(x_j) = f(x_j)$ on the Chebyshev grid $x_j = \cos(j\pi/N)$ for $j = 0, \ldots, N$.

The rationale for this choice of grid is that under the change of variable $x = \cos\theta$, the Chebyshev points become the equispaced samples $\theta_j = j\pi/N$. Unlike f, the function $g(\theta) = f(\cos\theta)$ is now 2π-periodic. Note that $g(\theta)$ inherits the smoothness of $f(x)$. The samples $g(\theta_j)$ can be made to cover the whole interval $[0, 2\pi]$ if we extend the range of j to be $0 \le j \le 2N-1$ (this corresponds to a mirror extension of the original samples.) The rationale for choosing Chebyshev polynomials is that $T_n(\cos\theta) = \cos(n\theta)$, so that Chebyshev interpolation of f from $f(x_j)$, with $0 \le j \le N-1$, is nothing but interpolation by trigonometric polynomials of g from $g(\theta_j)$, with $0 \le j \le 2N-1$.

This interpolant is built as follows. Start by submitting the $2N$ samples $g(\theta_j)$ to the discrete Fourier transform and back; this gives

$$g(\theta_j) = \sum_{n=-N}^{N-1} e^{in\theta_j} \tilde{g}_n.$$

The spectral interpolant $q(\theta)$ is built from these DFT coefficients as

$$q(\theta) = \sum_{n=-N}^{N}{}'' e^{in\theta} \tilde{g}_n, \tag{1}$$

where the double prime next to the sum indicates that the first and last terms are halved. This precaution is important to ensure that the interpolant of a real-valued function is real-valued.

The sum (1) reduces to the inverse discrete Fourier transform when $\theta = \theta_j$, so that $q(\theta_j) = g(\theta_j)$. Notice that g is even, so only the cosines are needed in this expansion:

$$q(\theta) = 2 \sum_{n=0}^{N}{}'' \cos(n\theta) \tilde{g}_n.$$

The Chebyshev interpolant of $f(x)$ is then simply $p(x) = q(\text{arccos}x)$. The coefficients are given explicitly as $c_n = 2\tilde{g}_n$ for $1 \leq n \leq N-1$, or $c_n = \tilde{g}_n$ for $n = 0, N$.
Spectral and Chebyshev interpolation methods are not only attractive because the FFT can be used to speed up computations, but because they have remarkable accuracy properties.

2 Spectral accuracy of Chebyshev interpolation

The first result concerns the algebraic decay of the interpolation error when f can be differentiated a finite number of times, or super-algebraic decay when f is infinitely differentiable.
We consider the native inner product for Chebyshev polynomials,

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \frac{dx}{\sqrt{1-x^2}},$$

with respect to which they are orthogonal. The associated weighted L^2_w norm

$$\|f\| = \left(\int_{-1}^{1} |f(x)|^2 \frac{dx}{\sqrt{1-x^2}} \right)^{1/2}$$

is used throughout this paper to measure the error. (The corresponding measure in $\theta = \arccos x$ is Lebesgue.) The related Sobolev spaces are

$$W^s_w = \{ f \in L^2_w \ : \ \|f\|^2_s = \sum_{k=0}^{s} \|f^{(k)}\|^2 < \infty \}.$$

The following result is elementary. The ideas can be traced back at least to [4]. A proof of the result as stated is in [6].

Theorem 1. *Let $f \in W^s_w$. Denote by p the N-point Chebyshev interpolant of f on $[-1, 1]$. Then*

$$\|f - p\| \leq C_s \, \|f\|_s \, N^{-s}.$$

In [6], Tadmor pushed the analysis further to obtain exponential decay in the case when f is real-analytic. A convenient setting is to assume that f extends analytically in the complex plane, in the "Bernstein" ellipse E_ρ with foci ± 1, center $z = 0$, and semi-axes

$$a_\rho = \frac{\rho + \rho^{-1}}{2}, \qquad b_\rho = \frac{\rho - \rho^{-1}}{2},$$

for some parameter $\rho > 1$ called the elliptical radius. Note that $a_\rho + b_\rho = \rho$. This ellipse has Cartesian equation

$$E_\rho = \{ z : \frac{(\text{Re } z)^2}{a_\rho^2} + \frac{(\text{Im } z)^2}{b_\rho^2} = 1 \},$$

and parametric equation

$$E_\rho = \{ z = \frac{\rho e^{i\theta} + \rho^{-1} e^{-i\theta}}{2} \ : \ \theta \in [0, 2\pi) \}.$$

Theorem 2 (Tadmor [6]). *Let f have an analytic extension in the open Bernstein ellipse E_{ρ_0} with elliptical radius $\rho_0 > 1$. For each $1 < \rho < \rho_0$, let*

$$M(\rho) = \max_{z \in E_\rho} |f(z)|.$$

Denote by p the N-point Chebyshev interpolant of f on $[-1, 1]$. Then for all $0 < \rho < \rho_0$,

$$\|f - p\| \le C \, \frac{M(\rho)}{\rho - \rho^{-1}} \, \rho^{-N}.$$

For the next result, which is possibly original, it is assumed instead that f is (Q, R) analytic, i.e., is real-analytic and obeys the smoothness condition

$$|f^{(n)}(x)| \le Q \, n! \, R^{-n}. \tag{2}$$

As noted in [5], p. 378, f obeys (2) for $x \in \mathbb{R}$ if and only if it can be analytically extended in the strip $|\text{Im } z| \le R$. This property holds because R is a lower bound on the convergence radius of the Taylor expansion of f at any point x. As a result it is a very natural class of analytic functions; Rudin denotes it by $C\{n!\}$.
We will only assume that f obeys (2) for $x \in [-1, 1]$, which results in a stadium-shaped analyticity region, as in Figure 1. Note that (Q, R) analyticity has already been used by the authors in [3, 2]. The main result is the following.

Theorem 3. *Let f be (Q, R)-analytic on $[-1, 1]$. Denote by p the N-point Chebyshev interpolant of f on $[-1, 1]$. Assume $N \ge 1/(2R)$. Then*

$$\|f - p\| \le C \, Q \, N \, \left[1 + \frac{1}{R^2} \right]^{1/4} \, \left[R + \sqrt{R^2 + 1} \right]^{-N}, \tag{3}$$

for some numerical constant $C > 0$.

A fortiori, the same bound holds for the weaker L^2 norm. The proof gives the value $\frac{5}{2}\sqrt{\frac{45e}{2}}$ for the numerical constant C; no attempt is made in this paper to find its sharp value. Note that $\left[R + \sqrt{R^2 + 1} \right]^{-N}$ corresponds to Tadmor's ρ^{-N}.
The error bound obeys the following asymptotic behaviors.

- As $R \to 0$, and if N less than or on the order of $1/R$, then the error bound is large.
- As $R \to 0$, and if $N \gg 1/R$, then the error bound is roughly proportional to $N R^{-1/2} e^{-RN}$.
- As $R \to \infty$, then the error bound is roughly proportional to $N (2R)^{-N}$.

3 Proof of Theorem 3

As mentioned in Section 1, f and p are respectively obtained from g and q through the change of variables $x = \cos\theta$. The factor $1/(\sqrt{1 - x^2})$ is precisely the Jacobian of this change of variables. Hence it suffices to prove that $\|g - q\|_2$ obeys the bound (3).

We start by listing the consequences of the smoothness condition (2). As is well-known, f has a unique analytic continuation as the Taylor series

$$f(z) = \sum_{n=0^\infty} \frac{f^{(n)}(x)}{n!} (z - x)^n,$$

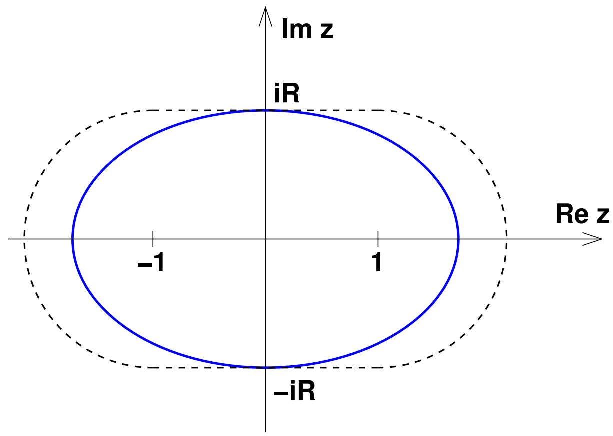

Figure 1: The stadium (dashed line) is the region of analyticity of f. The ellipse (blue, solid line) is the largest inscribed "Bernstein" ellipse with foci at ± 1.

which by (2) is manifestly convergent as soon as $|z - x| \le R$. Since $x \in [-1, 1]$, the domain of analyticity is the "stadium" illustrated in Figure 1, without its boundary. This shape is a subset of the strip $|\text{Im } z| < R$.
Furthermore, for all $x \in [-1, 1]$ we have the bound

$$\begin{aligned} |f(z)| &\le Q \sum_{n=0}^{\infty} \left(\frac{|z-x|}{R} \right)^n, \\ &\le \frac{Q}{1 - |z-x|R^{-1}}, \end{aligned}$$

which results in

$$|f(z)| \le \begin{cases} \frac{Q}{1-|z+1|R^{-1}} & \text{if Re } z < -1; \\ -\frac{Q}{1-|\text{Im} z|R^{-1}} & \text{if } -1 \le \text{Re } z \le 1; \\ \frac{Q}{1-|z-1|R^{-1}} & \text{if Re } z > 1 \end{cases} \tag{4}$$

The periodic function $g(\theta) = f(\cos\theta)$ also admits an analytic extension, best expressed through the function $h(z)$ such that $h(e^{i\theta}) = g(\theta)$. The result is the following lemma.

Lemma 1. *Let $h(e^{i\theta}) = f(\cos\theta)$, and assume that f is (Q, R)-analytic. Then h has a unique analytic continuation in the open annulus $|z| < R + \sqrt{R^2+1} < |z|^{-1}$, and obeys the bound*

$$|h(z)| \le \frac{Q}{1 - \frac{||z|-|z|^{-1}|}{2} R^{-1}}. \tag{5}$$

Proof of Lemma 1. The analytic extension $h(z)$ of $h(e^{i\theta})$ is related to $f(z)$ by the transformation

$$h(z) = f\left(\frac{z + z^{-1}}{2} \right). \tag{6}$$

Indeed, $h(e^{i\theta}) = f(\cos\theta)$, so the two expressions match when $|z| = 1$. There exists a neighborhood of $|z| = 1$ in which the right-hand side is obviously analytic, hence equal to $h(z)$ by uniqueness. The rationale for this formula is the fact that $\cos\theta = \cos(i \log e^{i\theta})$, and $(z + z^{-1})/2$ is just another expression for $\cos(i \log z)$.

More can be said about the range of analyticity of $h(z)$. The map $z \mapsto \zeta = (z + z^{-1})/2$ is a change from polar to elliptical coordinates [1]. It maps each circle $C_\rho = \{\rho e^{i\theta} : \theta \in [0, 2\pi)\}$ onto the ellipse E_ρ of parametric equation $\{(\rho e^{i\theta} + \rho^{-1} e^{-i\theta})/2 : \theta \in [0, 2\pi)\}$ introduced earlier. Notice that $|z| = \rho_0$ and $|z| = \rho_0^{-1}$ are mapped onto the same ellipse.
Figure 1 shows the open stadium of height $2R$ in which f is analytic, as well as the largest ellipse E_ρ inscribed in that stadium. Its parameter ρ obeys

$$|\rho - \rho^{-1}|/2 = R,$$

corresponding to the case $\theta = \pm\pi/2$. Solving for ρ, we get

$$\rho = R + \sqrt{R^2 + 1} \qquad \text{or} \qquad \rho = \frac{1}{R + \sqrt{R^2 + 1}}.$$

As a result, any z obeying $|z| < R + \sqrt{R^2 + 1} < |z|^{-1}$ corresponds to a point of analyticity of $f\left(\frac{z+z^{-1}}{2}\right)$, hence of $h(z)$.
To see why the bound (5) holds, substitute $\zeta = (z + z^{-1})/2$ for z in the right-hand-side of (4). The vertical lines $\text{Re}\ \zeta = \pm 1$ in the ζ plane become cubic curves with equations $(\rho + \rho^{-1}) \cos\theta = \pm 2$ in the z-plane, where $z = \rho e^{i\theta}$. Two regimes must be contrasted:

- In the region $|\text{Re}\ \zeta| \le 1$, we write

  $$|\text{Im}(z + z^{-1})| = |\rho \sin\theta - \rho^{-1} \sin\theta| \le |\rho - \rho^{-1}|,$$

  which leads to the bound (5) for h.

- Treating the region $\text{Re}\ \zeta > 1$ is only slightly more involved. It corresponds to the region $(\rho + \rho^{-1}) \cos\theta > 2$ in the z plane; we use this expression in the algebra below. We get

  $$\begin{aligned} |z + z^{-1} - 2| &= \left[\left((\rho + \rho^{-1}) \cos\theta - 2\right)^2 + (\rho - \rho^{-1})^2 \sin^2\theta\right]^{1/2} \\ &\le \left[\left((\rho + \rho^{-1}) \cos\theta - 2\cos\theta\right)^2 + (\rho - \rho^{-1})^2 \sin^2\theta\right]^{1/2}. \end{aligned}$$

  In order to conclude that (5) holds, this quantity should be less than or equal to $|\rho - \rho^{-1}|$. To this end, it suffices to show that

  $$(\rho + \rho^{-1} - 2)^2 \le (\rho - \rho^{-1})^2, \qquad \forall \rho > 0.$$

  Expanding the squares shows that the expression above reduces to $\rho + \rho^{-1} \ge 2$, which is obviously true.

- The region $\text{Re}\ \zeta < -1$ is treated in a very analogous manner, and also yields (5).

□

The accuracy of trigonometric interpolation is now a standard consequence of the decay of Fourier series coefficient of g. The result below uses the particular smoothness estimate obtained in Lemma 1. The proof technique is essentially borrowed from [6].

Lemma 2. *Let g be a real-analytic, 2π-periodic function of $\theta \in \mathbb{R}$. Define the function h of $z \in \{z : |z| = 1\}$ by $h(e^{i\theta}) = g(\theta)$, and assume that it extends analytically in the complex plane in the manner described by Lemma 1. Consider the trigonometric interpolant $q(\theta)$ of $g(\theta)$ from samples at $\theta_j = j\pi/N$, with $j = 0, \ldots, 2N-1$. Assume $N \geq 1/(2R)$. Then*

$$\|g - q\|_2 \leq C\; Q\; N\; \left[1 + \frac{1}{R^2}\right]^{1/4} \; \left[R + \sqrt{R^2+1}\right]^{-N}, \tag{7}$$

for some number $C > 0$.

Proof of Lemma 2. Write the Fourier series expansion of $g(\theta)$ as

$$g(\theta) = \sum_{n \in \mathbb{Z}} e^{in\theta} \hat{g}_n. \tag{8}$$

A comparison of formulas (8) and (1) shows that two sources of error must be dealt with:

- the truncation error, because the sum over n is finite in (1); and
- the aliasing error, because $\tilde{g}_n \neq \hat{g}_n$.

It is well-known that $\tilde{g}_n$ is a periodization of $\hat{g}_n$, in the sense that

$$\tilde{g}_n = \sum_{m \in \mathbb{Z}} \hat{g}_{n+2mN}.$$

This equation is (a variant of) the Poisson summation formula. As a result,

$$\|g - q\|_2^2 = \sum_{|n| \leq N}'' |\sum_{m \neq 0} \hat{g}_{n+2mN}|^2 + \sum_{|n| \geq N}'' |\hat{g}_n|^2. \tag{9}$$

The decay of $\hat{g}_n$ is quantified by considering that the Fourier series expansion of $g(\theta)$ is the restriction to $z = e^{i\theta}$ of the Laurent series

$$h(z) = \sum_{n \in \mathbb{Z}} \hat{g}_n z^n,$$

whereby the coefficients $\hat{g}_n$ are also given by the complex contour integrals

$$\hat{g}_n = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{h(z)}{z^{n+1}}\, dz. \tag{10}$$

This formulation offers the freedom of choosing the radius ρ of the circle over which the integral is carried out, as long as this circle is in the region of analyticity of $h(z)$.

Let us first consider the aliasing error – the first term in the right-hand side of (9). We follow [6] in writing

$$\begin{aligned}\sum_{m>0} \hat{g}_{n+2mN} &= \sum_{m>0} \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{h(z)}{z^{n+1+2mN}}\, dz, \\ &= \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{h(z)}{z^{n+1}(z^{2N}-1)}\, dz.\end{aligned}$$

For the last step, it suffices to take $\rho > 1$ to ensure convergence of the Neumann series. As a result,

$$|\sum_{m>0} \hat{g}_{n+2mN}| \leq \rho^{-n} \frac{1}{\rho^{2N}-1} \max_{|z|=\rho} |h(z)|, \qquad \rho > 1.$$

The exact same bound holds for the sum over $m < 0$ if we integrate over $|z| = \rho^{-1} < 1$ instead. Notice that the bound (5) on $h(z)$ is identical for ρ and ρ^{-1}.
Upon using (5) and summing over n, we obtain

$$\sum_{|n|\leq N}'' |\sum_{m\neq 0} \hat{g}_{n+2mN}|^2 \leq \left(\sum_{|n|\leq N}'' \rho^{2n}\right) \frac{4}{(\rho^{2N}-1)^2} \left[\frac{Q}{1-\frac{\rho-\rho^{-1}}{2}R^{-1}}\right]^2. \tag{11}$$

It is easy to show that the sum over n is majorized by $\rho^{2N}\frac{\rho+\rho^{-1}}{\rho-\rho^{-1}}$.
According to Lemma 1, the bound holds as long as $1 < \rho < R + \sqrt{R^2+1}$. The right-hand side in (11) will be minimized for a choice of ρ very close to the upper bound; a good approximation to the argument of the minimum is

$$\rho = \tilde{R} + \sqrt{\tilde{R}^2 + 1}, \qquad \tilde{R} = \frac{2N}{2N+1}R,$$

for which

$$\frac{1}{1-\frac{\rho-\rho^{-1}}{2}R^{-1}} = 2N+1.$$

The right-hand side in (11) is therefore bounded by

$$4Q^2(2N+1)\frac{1}{(\rho^N-\rho^{-N})^2}\frac{\rho+\rho^{-1}}{\rho-\rho^{-1}}.$$

This expression can be further simplified by noticing that

$$\rho^N - \rho^{-N} \geq \frac{1}{2}\rho^N$$

holds when N is sufficiently large, namely $N \geq 1/(2\log_2\rho)$. Observe that

$$\begin{aligned}\log_2\rho &= \frac{\ln\left(\tilde{R}+\sqrt{\tilde{R}^2+1}\right)}{\ln 2} \\ &= \frac{1}{\ln 2}\operatorname{arcsinh}(\tilde{R}) = \frac{1}{\ln 2}\operatorname{arcsinh}\left(\frac{2N}{2N+1}R\right),\end{aligned}$$

so the large-N condition can be rephrased as

$$R \geq \frac{2N+1}{2N}\sinh\left(\frac{\ln 2}{2N}\right).$$

It is easy to check (for instance numerically) that the right hand-side in this expression is always less than $1/(2N)$ as long as $N \geq 2$. Hence it is a stronger requirement on N and R to impose $R \geq 1/(2N)$, i.e., $N \geq 1/(2R)$, as in the wording of the lemma.
The resulting factor $4\rho^{-2N}$ can be further bounded in terms of R as follows:

$$\rho = \tilde{R} + \sqrt{\tilde{R}^2+1} \geq \left(\frac{2N+1}{2N}\right)[R+\sqrt{R^2+1}],$$

so

$$\rho^{-N} \leq \left(\frac{2N+1}{2N}\right)^{-N} [R + \sqrt{R^2+1}]^{-N}$$
$$\leq \left(\exp\frac{1}{2N}\right)^{-N} [R + \sqrt{R^2+1}]^{-N}$$
$$= \sqrt{e}\, [R + \sqrt{R^2+1}]^{-N}.$$

We also bound the factor $\frac{\rho+\rho^{-1}}{\rho-\rho^{-1}}$ – the eccentricity of the ellipse – in terms of R by following a similar sequence of steps:

$$\frac{\rho+\rho^{-1}}{\rho-\rho^{-1}} = \frac{2\sqrt{\tilde{R}^2+1}}{2\tilde{R}}$$
$$\leq \frac{2N+1}{2N}\sqrt{1+\frac{1}{R^2}}$$
$$\leq \frac{5}{4}\sqrt{1+\frac{1}{R^2}}.$$

After gathering the different factors, the bound (11) becomes

$$\sum_{|n|\leq N}{}'' \,|\sum_{m\neq 0} \hat{g}_{n+2mN}|^2 \leq 20\, e\, Q^2\, (2N+1)^2 \sqrt{1+\frac{1}{R^2}}\, \left[R+\sqrt{R^2+1}\right]^{-2N}. \tag{12}$$

We now switch to the analysis of the truncation error, i.e., the second term in (9). By the same type of argument as previously, individual coefficients are bounded as

$$|\hat{g}_n| \leq \left[\max(\rho,\rho^{-1})\right]^{-n} \frac{Q}{1-\frac{\rho-\rho^{-1}}{2}R^{-1}}.$$

The sum over n is decomposed into two contributions, for $n \geq N$ and $n \leq -N$. Both give rise to the same value,

$$\sum_{n\geq N} \rho^{-2n} = \frac{\rho^{-2N}}{1-\rho-2}.$$

We let ρ take on the same value as previously. Consequently, $\frac{Q}{1-\frac{\rho-\rho^{-1}}{2}R^{-1}} = 2N+1$, and, as previously,

$$\rho^{-2N} \leq e\,[R+\sqrt{R^2+1}]^{-2N}.$$

We also obtain

$$\frac{1}{1-\rho^{-2}} \leq \frac{\rho+\rho^{-1}}{\rho-\rho^{-1}} \leq \frac{5}{4}\sqrt{1+\frac{1}{R^2}}.$$

As a result, the overall bound is

$$\sum_{|n|\geq N} |\hat{g}_n|^2 \leq \frac{5}{2}\, e\, Q^2\, (2N+1)^2 \sqrt{1+\frac{1}{R^2}}\, \left[R+\sqrt{R^2+1}\right]^{-2N}. \tag{13}$$

We obtain (7) upon summing (12) and (13), and using $2N+1 \leq 5N/2$.

□

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