# A DDM double sweep preconditioner for the Helmholtz equation with matrix probing of the DtN map

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**Abstract**

We describe the structure of a fast solver for the Helmholtz equation in the optimized Schwarz framework, based on a preconditioner that leverages impedance-matching boundary conditions on subdomains. In the case of a simple 2D waveguide numerical example, the method requires no more than 4 GMRES iterations, independently of the frequency and the number of subdomains. The challenge remains to make each iteration fast: we give a partial answer to this question by showing how the Dirichlet to Neumann (DtN) map is accurately approximated in a compressed form via the recently introduced notion of matrix probing.

## Introduction

Domain Decomposition Methods (DDM) offer a very useful tool for numerically solving PDEs, but require additional ideas to operate optimally in the high-frequency regime [1]. The setting of our proposed method is a reformulation of the problem in terms of a set of unknown sources defined on artificial boundaries inside the domain. These sources must produce the same solution inside the individual subdomains, so a linear system must be solved to find them. It was shown in [2] that a Krylov method for this system can be set up, where the sources are defined from boundary conditions that match impedances of subdomains. The contributions of this paper are twofold: we propose 1) a preconditioner that takes advantage of the particular structure of the iteration matrix in the case of a layered partitioning ; and 2) an efficient computation of the DtN map using matrix probing.

## 1 Non-overlapping optimized Schwarz DDM

Consider a domain $\Omega$ with boundary $\partial\Omega$. We decompose $\Omega$ into $N$ non-overlapping slices $\Omega_{i,1\leq i\leq N}$, with artificial boundaries $\Sigma_{ij}$ between $\Omega_i$ and $\Omega_j$. (This is a layered partitioning, not a general 2D partitioning.) The iterative scheme, detailed in [2], uses impedance-matching boundary conditions on $\Sigma_{ij}$ and recasts the problem in terms of the set of interface data $g = \{g_{ij}, 1 \leq i \neq j \leq N, |i-j| = 1\}$. An iteration amounts to solving all subproblems in parallel:

$$\begin{aligned} -(\Delta + k^2)u_i^{(m+1)} &= 0 && \text{in } \Omega_i \\ (\partial_n + \mathcal{S})u_i^{(m+1)} &= (-\partial_n + \mathcal{S})u_j^{(m)} && \text{on } \Sigma_{ij} \\ &= g_{ij}^{(m)}, \end{aligned} \tag{1}$$

with $k$ the wavenumber and the update:

$$\begin{aligned} g_{ij}^{(m+1)} &= -\partial_n u_j^{(m+1)} + \mathcal{S}u_j^{(m+1)} && \text{on } \Sigma_{ij} \\ &= -g_{ji}^{(m)} + 2\mathcal{S}u_j^{(m+1)}. \end{aligned}$$

Boundary conditions on $\partial\Omega_i \cap \partial\Omega$ are conserved from the original problem. This procedure can be rewritten as a fixed point iteration on the unknowns $g$:

$$Fg = b, \tag{2}$$

where applying the operator $F$ amounts to solving the subproblems and updating $g$. The solution of problem (2) can be accelerated using GMRES.

The choice of operator $\mathcal{S}$ is critical for the rate of convergence. It was shown in [3] that the optimum is obtained if $\mathcal{S}$ is the DtN map of the Helmholtz operator on the corresponding interface.

## 2 Double sweep preconditioner

The matrix of the iteration operator $F$ (never formed in practice, but considered here for the purpose of analysis) consists of blocks arranged near the diagonal, each of which, when applied to a vector, amounts to solving a subproblem where the boundary source is restricted to one side of the domain. Provided that no reflection occurs at the opposite boundary, this matrix is easy to invert. An absence of reflection is hard to achieve in a numerical setting, but can be approached by using a sufficiently good Absorbing Boundary Condition (ABC).

The preconditioning strategy is to neglect those components of the iteration operator that are caused by spurious reflections, and would be zero in the absence of such reflections, to build an approximate inverse $\tilde{F}^{-1}$ of the iteration operator $F$.

| | $N=10$ | 20 | 50 | 100 | 200 | 400 |
|---|---|---|---|---|---|---|
| $k=10\pi$ | 3 | 3 | 4 | 4 | 4 | 4 |
| $40\pi$ | 3 | 3 | 3 | 3 | 4 | 4 |

Table 1: Convergence of the method with double sweep preconditioner and exact DtN map, applied to a rectangular waveguide with increasing number of subdomains $N$. The iteration count ($||r||_2/||r_0||_2 < 10^{-13}$) is small and independent of $N$ and wavenumber $k$.

By rearranging the terms of the matrix-vector product $g = \tilde{F}^{-1}r$, one can rewrite the preconditioner as a double sequence (forward and backward) of subproblems solutions, each problem taking into account the contribution of all its predecessors in the sequence. So, considering the forward sequence, the $i$-th component of $g$ is: $g_i = r_i + H_{f,i}^{-1} g_{i-1}$, with $H_{f,i}^{-1}$ the output of (1) for the $i$-th problem with impedance data $g_i$ on the left and 0 on the right, and starting with $g_1 = r_1$. Such a sequence of solves is called a sweep over the subdomains, hence the name of double sweep preconditioner.

The idea of sweeping was proposed in [4] for preconditioning the Helmholtz equation. A double sweep strategy was also proposed in [5]. These work use the sweeps to precondition the Helmholtz operator in the original domain, while we use them to precondition the operator $F$ of a Schwarz method. In addition, we precompute the DtN map by matrix probing rather than by using perfectly-matched layers (PML), leading to potential computational savings. Because the complexity of applying a probed DtN map hardly depends on its quality, we can use a better ABC (thicker layer) in our precomputation, hence potentially improved convergence properties for the DDM algorithm.

## 3 DtN map approximation via probing

We are thus looking for an accurate approximation $D$ to the DtN map $\mathcal{S}$ at some interface $\Sigma$. Consider the Helmholtz equation in a (PML) placed next to $\Sigma$. The operator $D$ is viewed as a black box that maps Dirichlet data on $\Sigma$ to the normal derivative, on $\Sigma$ as well, of the solution to the Helmholtz equation in the PML: $\partial_n \bar{u} = D\bar{u}$. We first precompute the matrix $D$ offline, then apply it to vectors on the fly as needed.

Matrix probing is used to make the precomputation of $D$ tractable. Suppose that we wish to approximate a matrix $D \in \mathbb{R}^{n\times n}$, but we only have access to a handful of products of $D$ with vectors. We assume $D$ can be written as a linear combination of a small number of basis matrices $B_j$, $D \approx \sum_{j=1}^{p} c_j B_j$ *fixed ahead of time*. Under various assumptions, notably $p \ll n$ (see [6] for details) we can recover the vector $c$ with great accuracy using only a few black box calls. For illustration, it is often advantageous to consider a single random vector $z$, so that $Dz \approx \sum_{j=1}^{p} c_j B_j z = \Psi_z c$, where the $B_j z$ are columns of $\Psi_z$. Solving for $c$ now requires the pseudo-inverse of $\Psi_z$, which can be quickly obtained since this is an $n \times p$ matrix with $p \ll n$.

Hence we need a relatively small set of basis matrices which can accurately approximate the DtN map $D$. There are different ways to do this: we can use a geometrical optics approximation with oscillations of the form $e^{i\omega\tau(x,y)}$ times a parametrized singular amplitude, see [7] for details, or else we can use the relaxed terms of the Padé expansion proposed in [2], obtained from a few 1D PDE solves. Both methods reproduce the numerical results presented earlier.

Current work focuses on further lowering the complexity of each subdomain solve $H^{-1}$ and each application of $D$. Preliminary results with a non-homogeneous medium tend to indicate that the behaviour of the method is not fundamentally changed.

## References


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