THIRD PROBLEM SESSION EXERCISES

The purpose of these exercises is to provide a proof of the Shannon Sampling Theorem discussed in class. We introduce a few extra concepts first. For a given $h > 0$, which will be fixed throughout, let $x_j = hj$ for all integers j. The semidiscrete Fourier transform (SFT) of a function u on $\mathbb{R}$ with sufficiently fast decay is defined to be

$$\hat{v}(k) = h \sum_{j=-\infty}^{\infty} e^{-ikx_j} u(x_j), \quad k \in [-\pi/h, \pi/h]$$

and the inverse semidiscrete Fourier transform (ISFT) is

$$u(x_j) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ikx_j} \hat{v}(k) \, \mathrm{d}k.$$

We will take as given that these two transforms are right and left inverses of one another if u decays fast enough and $\hat{v}$ is sufficiently regular, although this can be an extra exercise if you want to be complete. We emphasize that we will consistently use the notation $\hat{u}$ for the continuous Fourier transform of u, which is

$$\hat{u}(k) = \int_{\mathbb{R}} e^{-ixk} u(x) \, \mathrm{d}x,$$

while the SFT of u is $\hat{v}(k)$.

(1) Let

$$q(x) = \frac{\sin(\pi x/h)}{\pi x/h}.$$

Note that $q(x)$ is a smooth function. Let $\chi_{[-\pi/h,\pi/h]}$ be the indicator function of the interval $[-\pi/h, \pi/h]$. Show that the continuous Fourier transform of q is given by

$$\hat{q}(k) = h \; \chi_{[-\pi/h,\pi/h]}(k).$$

(2) For u decaying sufficiently fast, let $\hat{v}$ be the SFT of u. Show that

$$\chi_{[-\pi/h,\pi/h]}(k) \; \hat{v}(k)$$

is the continuous Fourier transform of

$$p(x) = \sum_{j=-\infty}^{\infty} u(x_j) q(x - x_j).$$

(3) For u sufficiently regular and decaying sufficiently fast, $\hat{u}$ the continuous Fourier transform of u, and $\hat{v}$ the SFT of u, prove the Poisson summation formula

$$\hat{v}(k) = \sum_{j=-\infty}^{\infty} \hat{u}(k + j\frac{2\pi}{h}), \quad k \in [-\pi/h, \pi/h].$$

Hint: Start by taking the ISFT.

(4) Combine the previous parts to prove the Shannon Sampling Theorem stated in class:

Theorem 1. *If u is band limited in the interval $[-\pi/h, \pi/h]$ (which means $\hat{u}$ is supported in that interval) and decays sufficiently fast at ∞ then*

$$u(x) = p(x)$$

where $p(x)$ is as defined in question (2). Also, if f and g both satisfy the same hypotheses then

$$\int_{\mathbb{R}} f(x)\ \overline{g(x)}\ \mathrm{dx} = h \sum_{j=-\infty}^{\infty} f(x_j)\ \overline{g(x_j)}.$$

When u is not band limited it is possible to use the Poisson summation formula to characterize the error incurred when approximating u by p in terms of the regularity of u. This error is known as aliasing and corresponds to the mistaken identification of higher frequency parts of the signal u as having lower frequencies because they are not sampled densely enough.