FIRST PROBLEM SESSION EXERCISES

(1) Suppose that an elastic string is stretched between two points at $x = 0$ and $x = L$ so that the magnitude of the tension $\underline{T}$ is a constant T. Also suppose that the mass density in the string is given by

$$\rho(x)\ \mathrm{d}x.$$

Now suppose that the string is perturbed slightly and begins to oscillate. Let $u(x,t)$ denote the transverse displacement of the oscillating string at position x and time t. For simplicity we assume we're working in 2D so there is only one transverse direction.

(a) Write the differential equation giving conservation of momentum of the moving string in terms of the displacement $u(x,t)$, the mass density ρ, and the transverse component of the tension $T_y(x)$.

(b) In order to obtain a closed system for $u(x,t)$ it is necessary to introduce another equation, usually called a constitutive relation, which gives a relationship between $u(x,t)$ and the transverse tension $T_y(x)$. Formulate such a relationship assuming that we can neglect changes in the magnitude of the tension (i.e. assume the magnitude of the tension remains constant). Combine the constitutive relation with part (a) to obtain a single differential equation satisfied by u.

(c) The equation in part (b) should be nonlinear. If the displacements are sufficiently small, then this equation can be linearized, and we get a good approximation of the behavior of the oscillating string. What is the linearized equation? What boundary conditions should we add to this equation?

(2) We define the Fourier transform for f a Schwartz function on $\mathbb{R}^n$ by

$$\mathcal{F}[f](\underline{k}) = \hat{f}(\underline{k}) = \int e^{-i\underline{x}\cdot\underline{k}} f(\underline{x})\ \mathrm{d}\underline{x}.$$

The inverse Fourier transform is then

$$\mathcal{F}^{-1}[f](\underline{x}) = \check{f}(\underline{x}) = \frac{1}{(2\pi)^n}\int e^{i\underline{x}\cdot\underline{k}} f(\underline{k})\ \mathrm{d}\underline{k}.$$

Show that if f is a real-valued Schwartz function then

$$\mathcal{F}[f(-\underline{x})](\underline{k}) = \overline{\mathcal{F}[f(\underline{x})]}(\underline{k}).$$

(3) For f and g Schwartz functions on $\mathbb{R}^n$ show that

$$(2\pi)^n \int f(\underline{x})\ \overline{g(\underline{x})}\ \mathrm{d}\underline{x} = \int \hat{f}(\underline{k})\ \overline{\hat{g}(\underline{k})}\ \mathrm{d}\underline{k}.$$

This is known as Parseval's Theorem. Use it to show that $\mathcal{F}$ can be extended to a continuous mapping from $L^2(\mathbb{R}^n)$ to itself. Can you write an explicit formula for this extension?

(4) Recall that the linear isotropic elastic wave equation is

$$\rho\, \partial_t^2 \underline{u} = \nabla\ (\ \lambda\ \nabla \cdot \underline{u}) + \nabla \cdot (\ \mu\ (\nabla \underline{u} + \nabla \underline{u}^T)\ )$$

where ρ is the density, λ and μ are the Lamé parameters and $\underline{u}$ is the displacement. Suppose that ρ, λ, and μ are all constants and let us look specifically for plane wave solutions of this equation of the form

$$\underline{u}(\underline{x}, t) = \psi(\underline{k} \cdot \underline{x} - ct)\ \underline{l}$$

where ψ is a C^2 function such that ψ'' is nonvanishing and $\underline{k}$, $\underline{l}$, and c are to be determined.

(a) Show that such $\underline{u}$ is a solution of the elastic wave equation given above if and only if

$$\rho c^2 \underline{l} = (\lambda + \mu)\ \underline{k}\ (\underline{k} \cdot \underline{l}) + \mu\ |\underline{k}|^2 \underline{l}.$$

(b) Suppose that $|\underline{k}| = 1$. Show $\underline{u}$ is a solution if and only if either

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

and $\underline{l}$ is parallel to $\underline{k}$ or

$$c = \sqrt{\frac{\mu}{\rho}}$$

and $\underline{l}$ is perpendicular to $\underline{k}$. In the first case the direction of propagation ($\underline{k}$) is parallel to the particle motion ($\underline{l}$) and so we have P-waves while in the second case the direction of propagation is perpendicular to the particle motion and so we have S-waves.