TAKEAWAYS FROM TALBOT 2018: THE MODEL INDEPENDENT THEORY OF ∞-CATEGORIES

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What is an ∞-category? It depends on who you ask:

- A category theorist: "It refers to some sort of weak infinite-dimensional category, perhaps an $(\infty, 1)$-category, or an (∞, n)-category, or an (∞, ∞)-category — or perhaps an internal or fibered version of one of these."
- Jacob Lurie: "It's a good nickname for an $(\infty, 1)$-category."
- A homotopy theorist: "I don't want to give a definition but if you push me I'll say it's a quasi-category (or maybe a complete Segal space)."

⋆ For us (interpolating between all of the usages above): "An ∞-*category* is a technical term meaning an object in some ∞-cosmos (and hence also an adjunction in its homotopy 2-category), examples of which include quasi-categories or complete Segal spaces; other models of $(\infty, 1)$-categories; certain models of (∞, n)-categories; at least one model of (∞, ∞)-categories; fibered and internal versions of all of the above; and other things besides.

The reason for this terminology is so that our theorem statements suggest their natural interpretation. For instance an *adjunction* between ∞-categories is defined to be an adjunction in the homotopy 2-category: this consists of a pair of ∞-categories A and B, a pair of ∞-functors $f : B \to A$ and $u : A \to B$, and a pair of ∞-natural transformations $\eta : \mathrm{id}_B \Rightarrow uf$ and $\epsilon : fu \Rightarrow \mathrm{id}_A$ satisfying the triangle equalities. Now any theorem about adjunctions in 2-categories specializes to a theorem about adjunctions between ∞-categories: e.g.

Proposition (equivalence-invariance of adjunctions)**.** *Given equivalences between ∞-categories $A \simeq A'$ and $B \simeq B'$, there exists a left adjoint to an ∞-functor $u : A \to B$ if and only if there exists a left adjoint to the equivalent ∞-functor $u' : A' \to B'$.*

Our main goal for the week is for every participant to be able to prove some theorems about ∞-categories.[1] This is feasible because all that is required is to prove theorems working in an arbitrary 2-category. This is the main takeaway:

> **Theorems proven in an arbitrary 2-category about appropriately-defined notions become theorems about ∞-categories. Consequently, the basic theory of ∞-categories does not have to be all that hard.**

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[1]For instance, you can really impress your friends by proving that right adjoints preserve limits without reading all the prerequisites for §5.2.3 of [L1] and §6.2 of [C].

1. Preview of coming attractions (Dominic Verity).

- The idea of an ∞-category is that it's some sort of category "weakly enriched" over spaces. This is made precise via *models* of ∞-categories which go by various names.
- ⋆ Our approach to developing ∞-category theory is *synthetic*, using the axiomatic framework of an "∞-cosmos" that describes the categories in which ∞-categories live as objects, rather than *analytic*, proving theorems using the combinatorics of any particular model.
- ⋆ We recast "∞-categories" as a technical term to mean the objects in any ∞-cosmos.
- Many flavors of categories (enriched, fibered, internal) live as objects in a 2-category whose 1-cells are the corresponding functors and whose 2-cells are the natural transformations, and the category theory of these objects can be developed internally to the 2-category.
- ⋆ Similarly, the ∞-categories in any ∞-cosmos are the objects of a 2-category whose 1-cells are the corresponding functors and whose 2-cells are the natural transformations.

2. ∞-Cosmoi and their homotopy 2-categories (Maru Sarazola).

> An ∞-*cosmos* axiomatizes the universe in which ∞-*categories* live as objects. We use the term "∞-category" very broadly to mean any structure to which category theory generalises in a homotopy coherent manner. Several models of $(\infty, 1)$-categories are ∞-categories in this sense, but our ∞-categories also include certain models of (∞, n)-categories as well as sliced versions of all of the above. This usage is meant to interpolate between the classical one, which refers to any variety of weak infinite-dimensional category, and the common one, which is often taken to mean quasi-categories or complete Segal spaces.
>
> This talk starts by reviewing the basic homotopy theory of quasi-categories and then introduces ∞-cosmoi, deduces some elementary consequences of their axioms, and constructs a *homotopy 2-category* associated with each one. We relate some common homotopical structures in the ∞-cosmos, such as homotopy equivalences and isofibrations, to their 2-categorical counterparts in its homotopy 2-category.
>
> References: [RV4, §2.1], [RVx, §1], [RV, chapter 1].

- Quasi-categories define a model of ∞-categories. Examples include topological spaces (via the total singular complex construction) and strict 1-categories (via the nerve construction).
- Any quasi-category has a *homotopy category* whose objects are the vertices, whose morphisms are homotopy classes of 1-simplices, and where composition is witnessed by the 2-simplices.
- An *equivalence* of quasi-categories is a "homotopy equivalence" defined using the interval $\mathbb{I}$ (the nerve of the free isomorphism). An *isofibration* between quasi-categories is a map with the right lifting property with respect to the inner horn inclusions and $\mathbb{1} \hookrightarrow \mathbb{I}$. A *trivial fibration* between quasi-categories is a map that is both an isofibration and an equivalence.
- ⋆ An ∞-cosmos is a "category of fibrant objects enriched over quasi-categories." It has objects, called ∞-*categories*, hom-spaces $\mathsf{Fun}(A, B)$, which are quasi-categories, and a specified class of maps called *isofibrations*. The first axiom asks for a bunch of limits to exist and satisfy a simplicially enriched universal property (expressed as an isomorphism of hom-spaces). The second axiom asks that various maps are isofibrations.
- A map $A \to B$ in an ∞-cosmos is called an *equivalence* if and only if $\mathsf{Fun}(X, A) \to \mathsf{Fun}(X, B)$ is an equivalence of quasi-categories for all X. These are equivalent to the "homotopy equivalences" defined using cotensors with the interval $\mathbb{I}$.

- ⋆ Any ∞-cosmos has a quotient 2-category called the *homotopy 2-category* whose objects are the ∞-categories in an ∞-cosmos, whose morphisms are the functors in the ∞-cosmos, and whose 2-cells we call natural transformations. Its hom-categories are defined to be the homotopy categories of the functor-spaces $\mathsf{Fun}(A, B)$.
- ⋆ There is a standard notion of equivalence in any 2-category. Importantly, these are exactly the same as the equivalences in the ∞-cosmos.

3. A menagerie of ∞-cosmological beasts (Joj Helfer).

An ∞-cosmos is not intended to axiomatise all of the ∞-category notions to be found in the literature; this talk will, however, establish that it does encompass very many of them. In particular we shall see that quasi-categories, complete Segal spaces, Segal categories, and naturally marked quasi-categories all define ∞-cosmoi. The objects in these models all deserve to be regarded as $(\infty, 1)$-categories.

This talk starts by reviewing the basic homotopy theory of quasi-categories. We shall also see that complete Segal objects in any well behaved Quillen model category provide a further example, and by iterating that observation we extend our observations to various models of (∞, n)-categories. Other higher examples discussed here include Θ_n-spaces and (weak) complicial sets.

In search of various *fibred* ∞-categorical notions, we introduce a slice construction for ∞-cosmoi and prove that every such slice is again an ∞-cosmos. Finally we relate the animals in our ∞-cosmos zoo by introducing a theory of ∞-cosmological *functors* and *biequivalences*.

References: [RV4, §2], [JT], [Re1], [V2], [B].

- ⋆ Quasi-categories define an ∞-cosmos whose objects are the quasi-categories, whose isofibrations are the isofibrations, whose equivalences are the equivalences, and with $\mathsf{Fun}(A, B) \coloneqq B^A$.
- ⋆ There is an ∞-cosmos whose objects are the regular 1-categories, whose isofibrations are the categorical isofibrations, whose equivalences are the categorical equivalences, and with $\mathsf{Fun}(A, B)$ defined to be the nerve of the category of functors from A to B.
- • *Segal categories* and *complete Segal spaces* each define model categories equipped with a Quillen equivalence to quasi-categories. These functors satisfy certain conditions that can be used to make the model categories enriched over the model structure for quasi-categories. Hence Segal categories (that are Reedy fibrant) and complete Segal spaces each define an ∞-cosmos.
- • For any ∞-cosmos $\mathscr{K}$ and any ∞-category $B \in \mathscr{K}$, there is an ∞-cosmos $\mathscr{K}_{/B}$ whose objects are isofibrations over B.
- • A *cosmological functor* between ∞-cosmoi is a simplicially enriched functor $\mathscr{K} \to \mathscr{L}$ that preserves the isofibrations and all of the limits. Examples include pullback $f^* : \mathscr{K}_{/B} \to \mathscr{K}_{/A}$ along $f : A \to B$, $\mathsf{Fun}(X, -) : \mathscr{K} \to q\mathscr{C}at$, and right Quillen functors.
- • Right Quillen equivalences define cosmological functors that are additionally *biequivalences* (about more which later).

4. Adjunctions, limits, and colimits in homotopy 2-categories (Emma Phillips).

We have already seen that some 2-categorical notions may be imported into the world of ∞-cosmoi directly from their associated homotopy 2-categories. In this talk we continue on that journey, applying this insight to develop a theory of *adjunctions* between ∞-categories in an ∞-cosmos.

We start by reviewing the theory of (equationally defined) adjunctions and *mates* in 2-categories and discussing their generalisation to a theory of *absolute lifting diagrams*. We apply these notions in the homotopy 2-category of an ∞-cosmos, and derive some elementary consequences. These observations lead us to a discussion of the *internal* limits and colimits that can live within an ∞-category, and we give elementary proofs of familiar results such as the preservation of limits by right adjoints.

References: [RV1, §4-5], [KS], [RV, chapter 2].

- ⋆ An **adjunction** between ∞-categories is defined by specializing the general notion of an adjunction in any 2-category to the homotopy 2-category of ∞-categories, ∞-functors, and ∞-natural transformations.
- ⋆ Now any 2-categorical theorem about adjunctions becomes a theorem about ∞-categories, for instance:
  - Adjunction compose.
  - The left adjoint to a given functor is unique up to natural isomorphism.
  - Any equivalence can be promoted to an adjoint equivalence.
- ⋆ The counit $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ of an adjunction defines an *absolute right lifting* of the functor $\mathrm{id}_A\colon A \to A$ through $f : B \to A$, meaning that any 2-cell as below-left factors uniquely as below-right.

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] \arrow[d, phantom, shift left=4, "\Downarrow\exists!\beta"] & B \arrow[d, "f"] \arrow[d, phantom, shift right=4, "\Downarrow\epsilon"] \\ A \arrow[r, equal] \arrow[ur, "u"] & A \end{tikzcd}$$

  That is to say, there is a bijection between $\alpha : fb \Rightarrow a$ and $\beta : b \Rightarrow ua$ implemented by pasting with the counit.
- ⋆ For a simplicial set J and ∞-category A, the cotensor A^J is the ∞-*category of diagrams of shape J in A*. The ∞-category *A admits all limits of shape J* if the constant diagram functor $\Delta\colon A \to A^J$ admits a right adjoint. The counit defines the *limit cone*.
- ⋆ Generalizing the above, a *limit* of a diagram $d : 1 \to A^J$ of shape J in A is an absolute right lifting of this functor through $\Delta : A \to A^J$.
- ⋆ Using a 2-categorical version of the classical proof, you can show that right adjoints preserve limits (Exercise 2.4.iii).
- ⋆ As a corollary, equivalences preserve limits and colimits since equivalences are both left and right adjoints.

Tuesday, May 28

Yesterday we discovered that by magic:

- ⋆ We get a strict 2-category, called the *homotopy 2-category*, whose objects are ∞-categories, whose morphisms are ∞-functors, and whose 2-cells are ∞-natural transformations.
- ⋆ Moreover, the equivalences in this 2-category — given by a pair of ∞-categories, an opposing pair of functors, and a pair of natural isomorphisms relating the composites to identities — correspond to the commonly accepted notion of equivalence between ∞-categories in each model.

The upshot of this result is that:

Working up to the standard 2-categorical notion of equivalence in the homotopy 2-category is "homotopically correct."

This gives us confidence to use 2-categorical techniques to develop the theory of ∞-categories. For example:

- We defined an adjunction between ∞-categories to be an adjunction in the homotopy 2-category. Now all theorems about adjunctions in 2-categories become theorems about adjunctions between ∞-categories.
- In the homotopy 2-category we are entitled to form an ∞-category A^J of diagrams valued in an ∞-category A indexed by a simplicial set J. We can now define limit and colimit functors as adjoints:

$$\begin{tikzcd} A \arrow[r, "\Delta"] & A^J \arrow[l, bend right=50, "\mathrm{colim}"'] \arrow[l, bend left=50, "\lim"] \arrow[l, phantom, shift right=2, "\perp"] \arrow[l, phantom, shift left=2, "\perp"] \end{tikzcd}$$

  and prove a cheap version of "right adjoints preserve limits" simply because adjunctions compose and adjoints are unique.

But where are the universal properties? Today we'll see the answer.

5. Arrow and comma ∞-categories (Laura Wells).

In classical category theory, the equational account of adjunctions provides only one promontory from which to survey the world of universal constructions. For many purposes, notions of *representability* play an equally important role, and in abstract category theory these are often expressed in the language of *modules* (sometimes called *profunctors* or *correspondences*) between categories.

One well worn route to a theory of modules in traditional (internal) category theory is to study the abstract properties of *comma categories*. In 2-category theory these generalise to 2-dimensional limit structures called *comma objects*, and we review their theory with a view to re-interpreting such notions within the theory of ∞-cosmoi. We show that any ∞-cosmos admits the construction of the comma ∞-category associated to any cospan that possesses a homotopically well-behaved and simplicially enriched variant of the 2-universal property enjoyed by comma objects.

Our hope is to simplify some computations involving comma objects by executing them within the homotopy 2-category associated with our ∞-cosmos, and this leads us to investigating their universal properties in there. In doing so we discover that they only satisfy a certain *weak 2-universal* property, which we establish and apply.

References: [RV1, §3], [RV4, §3], [RV, chapter 3].

- ⋆ For any functors between 1-categories $C \xrightarrow{g} A \xleftarrow{f} B$, there is a *comma category* $f \downarrow g$ whose
  - objects are triples $(b \in B, c \in C, \alpha : fb \to gc \in A)$
  - morphisms from $(b, c, \alpha : fb \to gc)$ to $(b', c', \alpha' : fb' \to gc')$ are pairs $(\beta : b \to b' \in B, \gamma : c \to c' \in C)$ so that the square commutes in A

  The aim is to generalize this construction to ∞-categories.
- You can define the notion of a *comma object* in any 2-category to be the object that represents the comma category construction in $\mathcal{C}at$. However, this universal property is too strict to characterize the comma ∞-category in the homotopy 2-category.

- ⋆ To explain, let A be a quasi-category and write $\mathbb{2}$ for $\Delta[1]$. There is a canonical functor of 1-categories
$$\mathrm{ho}(A^{\mathbb{2}}) \to (\mathrm{ho}\, A)^{\mathbb{2}}$$
from the homotopy category of the quasi-category of arrows to the category of arrows in the homotopy category. This functor is not an isomorphism or equivalence but instead is *smothering*: surjective on objects, full, and conservative.
- ⋆ The *comma ∞-category* of $C \xrightarrow{g} A \xleftarrow{f} B$ is constructed by the pullback
$$\begin{tikzcd} f \downarrow g \arrow[r, "\phi"] \arrow[d, two heads, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d, two heads, "{(p_1,p_0)}"] \\ C \times B \arrow[r, "g\times f"'] & A \times A \end{tikzcd}$$
in the ∞-cosmos $\mathcal{K}$. In the homotopy 2-category $\mathfrak{h}\mathcal{K}$, the maps $(p_1, p_0) : f \downarrow g \twoheadrightarrow C \times B$ define the codomain and domain projections. The map $\phi : f \downarrow g \to A^{\mathbb{2}}$ represents a 2-cell
$$\begin{tikzcd} & f \downarrow g \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\phi}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$
- In $\mathcal{K}$, $f \downarrow g$ has a strict universal property given by the pullback. In $\mathfrak{h}\mathcal{K}$, $f \downarrow g$ has a weak universal property given by three operations we call *1-cell induction*, *2-cell induction*, and *2-cell conservativity*.
- ⋆ The upshot of this weak universal property is that whiskering a functor a as below right with the 2-cell ϕ above induces a bijection between 2-cells in the homotopy 2-category as below-left and fibered isomorphism classes of maps of spans as below-right:
$$\left\{\begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\alpha}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}\right\} \quad \leftrightsquigarrow \quad \left\{\begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, "a"] & \\ C & & B \\ & f \downarrow g \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] & \end{tikzcd}\right\}_{/\cong}$$
- Finally, the weak universal property of the comma ∞-category carries $f \downarrow g \twoheadrightarrow C \times B$ up to *fibered equivalence over* $C \times B$, i.e., up to equivalence in $\mathcal{K}_{/C\times B}$.

6. The universal properties of adjunctions, limits, and colimits (Lyne Moser).

In this talk, we present a variety of results that describe the universal properties of adjunctions, limits, and colimits. A general theme is that such universal properties can be described by fibered equivalences between comma ∞-categories. For example, a functor $u : A \to B$ in an ∞-cosmos $\mathcal{K}$ has a left adjoint if and only if the associated comma $B \downarrow u$ is represented by some $f : B \to A$, in the sense that $B \downarrow u \simeq f \downarrow A$ in the sliced ∞-cosmos $\mathcal{K}_{/A\times B}$. This fibered equivalence pulls back to define an equivalences between the internal mapping spaces of A and B.

Comma ∞-categories can also be used to define the ∞-category of cones above or below a fixed or varying diagram. A limit of a diagram d is then an element that represents this comma ∞-category of cones over d and the limit defines a terminal element in this ∞-category of cones. For diagrams indexed by a simplicial set J, the limit cone can also be understood as the right Kan extension to a diagram indexed by the simplicial set $J^{\triangleleft}$ which has a cone point adjoined above the diagram. Specializing this result and its dual to the case of pullbacks and pushouts allows us to define the loops $\vdash$ suspension adjunction in any pointed ∞-category.

References: [RV1, §4-5], [RV, §3.4-5, chapter 4].

- ⋆ In $\mathscr{C}at$, $f \dashv u$ iff $A(fb, a) \cong B(b, ua)$ naturally in a and b iff there is an isomorphism of comma categories $f \downarrow A \cong B \downarrow u$ over $A \times B$ iff there is an equivalence of comma categories $f \downarrow A \simeq B \downarrow u$ over $A \times B$ (since any fibered equivalence of this kind is necessarily an isomorphism).
- ⋆ In $\mathscr{C}at$, $d : J \to A$ has a limit $\ell \in A$ iff $A(a, \ell) \cong A^J(\Delta a, d)$ naturally in a iff there is an isomorphism/equivalence of comma categories $A \downarrow \ell \simeq \Delta \downarrow d$ over A.
- • Both of these results generalize to ∞-categories as we now explain. Recall $f \dashv u$ or $d : 1 \to A^J$ admits a limit $\ell : 1 \to A$ iff there are absolute right liftings

$$\begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon"] & A \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon"] & A^J \end{tikzcd}$$

- • By Theorem 3.4.7 of [RV], a 2-cell as below-left

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A \end{tikzcd}$$

  is an absolute right lifting if and only if it induces a fibered equivalence $B \downarrow \ell \simeq f \downarrow g$ over $C \times B$.
- ⋆ Specializing to the absolute right lifting diagrams that characterize adjoints and limits we have $f \dashv u$ iff $f \downarrow A \simeq B \downarrow u$ over $A \times B$ and $\ell : 1 \to A$ is the limit of $d : 1 \to A^J$ iff $A \downarrow \ell \simeq \Delta \downarrow d$ over A.
- • An element $t : 1 \to A$ is *terminal* iff this functor is right adjoint to $! : A \to 1$. As a corollary of the second facto above, t is terminal iff the domain projection $p_0 : A \downarrow t \twoheadrightarrow A$ is a trivial fibration, since $A \downarrow t \simeq ! \downarrow 1 \cong A$ over A.
- ⋆ For $J \in s\mathscr{S}et$, define $J^{\triangleright} = J \star \mathbb{1}$ and $J^{\triangleleft} = \mathbb{1} \star J$. The idea is that $J^{\triangleleft}$ has a canonical inclusion $J \hookrightarrow J^{\triangleleft}$ and a cone point $\top$. Then by more general nonsense about absolute right lifting diagrams, A has all limits of shape J if and only if there exists a right adjoint as below-left and A has all colimits of shape J if and only if there exists a left adjoint as below-right

$$\begin{tikzcd} A^{J^{\triangleleft}} \arrow[r, bend left, "\mathrm{res}"] \arrow[r, phantom, "\perp"] & A^J \arrow[l, bend left, dashed, "\mathrm{ran}"] \end{tikzcd} \qquad \begin{tikzcd} A^{J^{\triangleright}} \arrow[r, bend right, "\mathrm{res}"'] \arrow[r, phantom, "\perp"] & A^J \arrow[l, bend right, dashed, "\mathrm{lan}"'] \end{tikzcd}$$

- • Specializing J to $\lrcorner := \Lambda^2[2]$ or $\ulcorner := \Lambda^0[2]$ we can define *pullback* and *pushout squares*.
- • An ∞-category A is *pointed* if it has an element $* : 1 \to A$ that is both initial and terminal. In this case, you can define a functor $\bar{\rho} : A \to A^{\lrcorner}$ that sends an element $a : 1 \to A$ to the cospan

$* \to a \leftarrow *$. The *loops functor* is defined to be the limit of this family of diagrams

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ A \arrow[ur, "\Omega"] \arrow[r, "\bar{\rho}"'] \arrow[ur, phantom, shift left=2, "\Downarrow"] & A^{\lrcorner} \end{tikzcd}$$

The suspension functor Σ is defined dually to be the colimit of a diagram $A \to A^{\ulcorner}$.

- If A has all pullbacks and pushouts, then the lan $\dashv$ res and res $\dashv$ ran adjunctions for pullbacks and pushouts pullback to define a loops suspension adjunction

$$\begin{tikzcd} A \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

7. Homotopy coherent adjunctions and monads (Martina Rovelli).

In this talk, we discover how the 2-categorically defined adjunctions discussed in the last two talks extend, in a homotopically unique way, to give *homotopy coherent adjunctions*. These structures encapsulate all of the higher coherence data that one would hope for from a fully homotopical adjunction and they are realised as simplicial functors $\mathscr{A}dj \to \mathscr{K}$ mapping a certain combinatorially defined simplicial category $\mathscr{A}dj$ into our ∞-cosmos $\mathscr{K}$. It comes as somewhat of a surprise to discover that this simplicial category $\mathscr{A}dj$ is actually no-more-nor-less than the 2-category long dubbed the *generic adjunction* by 2-category theorists. The proofs of these results will take us deep into the homotopy theoretic weeds of simplicial computads, local horn extension arguments, and the rather quaint theory of *squiggles* used to describe the simplices in the hom-spaces of $\mathscr{A}dj$.

Any self-respecting adjunction of ∞-categories should give rise to a monad, those in turn should admit the construction of monadic adjunctions. While the 2-categorical notion of an adjunction encodes a defining universal property that allows us to extend them to fully homotopy coherent structures, it would be naïve of us to hope that monads of ∞-categories might be captured simply as 2-categorical monads in homotopy 2-categories. Since monads are purely equational beasts that possess no corresponding universal property from which to derive higher coherence data, all of this must be given explicitly.

We round out this discussion by defining *homotopy coherent monads*, or simply just *∞-monads*, to be simplicial functors $\mathscr{M}nd \to \mathscr{K}$, where $\mathscr{M}nd$ is the simplicial full subcategory of $\mathscr{A}dj$ spanning one of its objects. Now we see that our adjunctions extend to homotopy coherent structures, parameterised by $\mathscr{A}dj$, which themselves restrict to give homotopy coherent monads, parameterised by $\mathscr{M}nd$, just as we had hoped. Concretely the simplicial category $\mathscr{M}nd$ has the nerve of the algebraist's ordinal category $\mathbf{\Delta}_+$ as its unique endo-hom-space and ordinal sum as its composition.

References: [RV2], [SS], [RV, chapter 8].

★ The idea of a *homotopy coherent adjunction* or a *homotopy coherent monad* is that it's an extension of the data of an adjunction or monad (which involve objects, morphisms, and morphisms between morphisms) to an include higher-dimensional coherence morphisms in all dimensions. These are structures that live in a quasi-categorically enriched category.

- In a 2-category like $\mathfrak{h}\mathscr{K}$, you have objects A, B and then hom-categories $\operatorname{ho}\mathsf{Fun}(A, B)$ whose objects we write as $f\colon A \to B$ and whose arrows we write as $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$. The arrows satisfy composition relations, maybe $\beta \cdot \alpha = \gamma$.

★ In a quasi-categorically enriched category like $\mathscr{K}$, you have objects A, B and hom-quasi-categories $\mathsf{Fun}(A, B)$ whose objects we write as $f\colon A \to B$ but also think of as the vertices $f \in \mathsf{Fun}(A, B)$; we might call these *0-arrows* from A to B. The 1-simplices, or *1-arrows*, we can write as $\alpha\colon f \to g$. Each 1-arrow represents a 2-cell as above in the homotopy 2-category. If $\beta\colon g \to h$ and $\gamma\colon f \to h$ are so that $\beta \cdot \alpha = \gamma$ in the homotopy 2-category, the reason this is so is because there exists a 2-simplex in $\mathsf{Fun}(A, B)$ with boundary

$$\begin{tikzcd} & g \arrow[dr, "\beta"] & \\ f \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & h \end{tikzcd}$$

The upshot is that equations between natural transformations in the homotopy 2-category are witnessed by 2-simplices in $\mathsf{Fun}(A, B)$.

- There is a free 2-category $\mathscr{A}dj$ containing an adjunction in the sense that 2-functors $\mathscr{A}dj \to \mathscr{C}$ correspond bijectively to adjunctions in the 2-category $\mathscr{C}$. It has two objects and four hom-categories that we picture via the cartoon:

$$\begin{tikzcd} \mathbf{\Delta}_+ \arrow[loop left] & + \arrow[rr, bend left, "\mathbf{\Delta}_\perp \cong \mathbf{\Delta}_\top{}^{\mathrm{op}}"] \arrow[rr, phantom, "\perp"] & & - \arrow[ll, bend left, "\mathbf{\Delta}_\top \cong \mathbf{\Delta}_\perp{}^{\mathrm{op}}"] & \mathbf{\Delta}_+^{\mathrm{op}} \arrow[loop right] \end{tikzcd}$$

- A *monad* in a 2-category consists of an object B, a map $t\colon B \to B$, and 2-cells

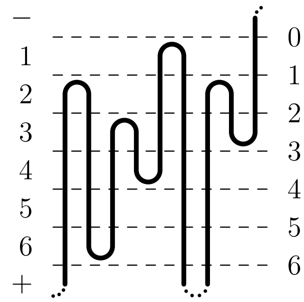
$$\begin{tikzcd} B \arrow[r, bend left, equal] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\eta"] & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "t^2"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\mu"] & B \end{tikzcd}$$

satisfying relations: an associativity square and two unit triangle. The full subcategory $\mathscr{M}nd$ of $\mathscr{A}dj$ at $+$ is the free 2-category containing a monad: 2-functors $\mathscr{M}nd \to \mathscr{C}$ correspond bijectively to monads in the 2-category $\mathscr{C}$.

★ There is a way to make a 2-category into a simplicial category with the same objects and whose hom-spaces are the nerves of the hom-categories. In particular, the 2-categories $\mathscr{A}dj$ and $\mathscr{M}nd$ can be thought of as simplicial (in fact quasi-categorically enriched) categories.

★ Inspired by the universal properties above define a *homotopy coherent adjunction* $\mathscr{A}dj \to \mathscr{K}$ and a *homotopy coherent monad* to be a simplicial functor $\mathscr{M}nd \to \mathscr{K}$.

- To explain what this definition entails, we present the simplicial category $\mathscr{A}dj$ in a different way. It has the same two objects $+$ and $-$ and four hom-simplicial sets $\mathscr{A}dj(+,+)$, $\mathscr{A}dj(+,-)$, $\mathscr{A}dj(-,+)$, and $\mathscr{A}dj(-,-)$. The n-simplices in $\mathscr{A}dj(-,+)$ are *strictly undulating squiggles on $n+1$-lines from $-$ to $+$*, eg when $n = 6$:



The faces of this simplex are given by removing the appropriate line and then possibly stretching to smooth out the diagram. The degeneracies are given by duplicating the appropriate line. The composition is given by horizontal concatenation.

- As an exercise, you can write down 0-, 1-, and 2-dimensional simplices in $\mathscr{A}dj$ that define the functors, unit, and counit, and witnesses for the triangle identities of an adjunction.

$\star$ Now a homotopy coherent adjunction $\mathbb{A} : \mathscr{A}dj \to \mathscr{K}$ maps all of the data just enumerated to corresponding data in $\mathscr{K}$. To define an adjunction in $\mathfrak{h}\mathscr{K}$ you just need to give $(A, B, f, u, \eta, \epsilon)$. To define a homotopy coherent adjunction in $\mathscr{K}$, you need to give these things (two objects, two 0-arrows, and two 1-arrows) but also specify the 2-arrows witnessing the triangle identities, and various 3-arrows, and various 4-arrows, etc as encoded by the squiggle diagrams all the way up.

$\star$ Surprisingly: any adjunction in $\mathfrak{h}\mathscr{K}$ can be extended to a homotopy coherent adjunction in $\mathscr{K}$.

$\star$ This is not true for monads: a monad in $\mathfrak{h}\mathscr{K}$ cannot necessarily be extended to a homotopy coherent monad in $\mathscr{K}$.

8. Homotopy coherent monadicity and descent (Kyle Ferendo).

Continuing the narrative of the last talk, we show how to derive an ∞-cosmological *Eilenberg-Moore object*, constructing the ∞-*category of algebras* associated to each ∞-monad. We prove a variant of the *Beck monadicity theorem* and examine a few applications to homotopy coherent algebra, ∞-category theory, and higher descent theory.

This talk commences with a review of the rubric of *weighted limits and colimits* in enriched category theory. Proceeding by analogy with classical 2-categorical accounts, we define the Eilenberg-Moore object of an ∞-monad to be a certain *flexibly weighted* simplicial limit. We briefly examine some of the properties of these flexible limits, including the key fact that the flexible limit of a diagram of ∞-categories that admit (and whose connecting maps preserve) a class of (co)limits again admits such (co)limits.

We derive an adjunction between the Eilenberg-Moore object of an ∞-monad and its underlying object, showing that the ∞-monad associated with this adjunction is simply just the ∞-monad we started with. This is the *monadic adjunction* characterized by Beck's theorem.

Now we consider an arbitrary adjunction and construct a comparison between its domain and the Eilenberg-Moore object of its associated ∞-monad. After a brief discussion of (split) simplicial objects in ∞-categories and their realisations, we examine some consequences of properties of that comparison and prove an

∞-categorical re-imagining of Beck's monadicity and descent theorems. Our primary goal here shall be to illustrate the way in which the proofs of these results proceed as direct analogues of their traditional 1-categorical counterparts.

References: [RV2, §5-7], [RV3], [Su], [Z], [RV, chapter 9].

- ⋆ Given a monad $T = (t, \eta, \mu)$ on a 1-category B, there exists an *Eilenberg-Moore category* B^t of *T-algebras* which is equipped with an adjunction

$$\begin{tikzcd} B \arrow[r, bend left, "f^t"] \arrow[r, phantom, "\perp"] & B^t \arrow[l, bend left, "u^t"] \end{tikzcd}$$

that is the terminal adjunction whose underlying monad is $T = (t, \eta, \mu)$. In particular, if given another adjunction

$$\begin{tikzcd} B \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u"] \end{tikzcd}$$

with the same underlying monad there exists a canonical functor $k : A \to B^t$ that commutes with everything in sight.
- • To build the analogous ∞-category of algebras for a homotopy coherent monad, we would like to form the simplicially enriched right Kan extension

$$\begin{tikzcd} \mathscr{M}nd \arrow[rr, "\mathbb{T}"] \arrow[dr, hook] & \arrow[d, phantom, "\Uparrow\cong"] & \mathscr{K} \\ & \mathscr{A}dj \arrow[ur, dashed, "\mathbb{A}^{\mathbb{T}}"'] & \end{tikzcd}$$

But does this exist in any ∞-cosmos $\mathscr{K}$? It turns out the answer is yes and detour will explain why.
- ⋆ Given a pair of simplicial functors $F : \mathscr{D} \to \mathscr{K}$ and $W : \mathscr{D} \to s\mathscr{S}et$, the *limit of the diagram F weighted by W* is an object $\{W, F\}_{\mathscr{D}} \in \mathscr{K}$ so that there is an isomorphism of simplicial sets

$$\mathsf{Fun}(X, \{W, F\}_{\mathscr{D}}) \cong s\mathscr{S}et^{\mathscr{D}}(W, \mathsf{Fun}(X, F)).$$

For example:
  - – If $W : \mathscr{D} \to s\mathscr{S}et$ is the functor constant at the terminal object $\mathbb{1} = \Delta[0] \in s\mathscr{S}et$, the weighted limit $\{\mathbb{1}, F\}_{\mathscr{D}}$ is the (conical) simplicially enriched limit discussed in the definition of an ∞-cosmos.
  - – If $\mathscr{D} = \mathbb{1}$ is the category with one object and only its identity arrow, then both the weight and diagram define objects $W \in s\mathscr{S}et$ and $F \in \mathscr{K}$ and the weighted limit $\{W, F\}_{\mathbb{1}}$ is the simplicial cotensor F^W also discussed in the definition of an ∞-cosmos.
  - – If the weight $W = \mathscr{D}(d, -) : \mathscr{D} \to s\mathscr{S}et$ is representable, then the weighted limit

$$\{\mathscr{D}(d, -), F\}_{\mathbb{1}} \cong F(d)$$

by the Yoneda lemma.
- ⋆ As a consequence of the limit axiom for an ∞-cosmos $\mathscr{K}$, the simplicially enriched category $\mathscr{K}$ admits all weighted limits whose weights $W \in s\mathscr{S}et^{\mathscr{D}}$ are *flexible* aka *projectively cofibrant*, meaning cofibrant objects in the projective model structure on $s\mathscr{S}et^{\mathscr{D}}$ relative to any model structure on $s\mathscr{S}et$ whose cofibrations are the monomorphisms.

- The free homotopy coherent adjunction $\mathcal{A}dj$ is a cofibrant simplicial category aka a *simplicial computad*. Because of this, many weights built from $\mathcal{A}dj$ turn out to be flexible.
- In particular, we can build the homotopy coherent adjunction $\mathbb{A}^{\mathbb{T}} : \mathcal{A}dj \to \mathcal{K}$ defined by right Kan extending the homotopy coherent monad $\mathbb{T} : \mathcal{M}nd \to \mathcal{K}$ acting on $B \in \mathcal{K}$ via a flexible weighted limit where the weights are taken to be the restrictions of the representable functors $\mathcal{A}dj_+, \mathcal{A}dj_- \in s\mathcal{S}et^{\mathcal{A}dj}$ along $\mathcal{M}nd \hookrightarrow \mathcal{A}dj$:

$$\begin{tikzcd}
\mathcal{A}dj \cong (\mathcal{A}dj^{\mathrm{op}})^{\mathrm{op}} \arrow[r, "\mathcal{Y}"] & (s\mathcal{S}et^{\mathcal{A}dj})^{\mathrm{op}} \arrow[r, "\mathrm{res}"] & (s\mathcal{S}et^{\mathcal{M}nd})^{\mathrm{op}} \arrow[r, "\{-,\mathbb{T}\}_{\mathcal{M}nd}"] & \mathcal{K} \\
+ \arrow[r, mapsto] & \mathcal{A}dj_+ \arrow[r, mapsto] & \mathrm{res}\,\mathcal{A}dj_+ \cong \mathcal{M}nd_+ \arrow[r, mapsto] & \{\mathcal{M}nd_+, \mathbb{T}\} \cong B \\
- \arrow[r, mapsto] & \mathcal{A}dj_- \arrow[r, mapsto] & \mathrm{res}\,\mathcal{A}dj_- \arrow[r, mapsto] & \{\mathrm{res}\,\mathcal{A}dj_-, \mathbb{T}\} =: B^t
\end{tikzcd}$$

  This weighted limit defines the ∞-*category of* $\mathbb{T}$-*algebras in* B together with a homotopy coherent adjunction

$$\begin{tikzcd}
B \arrow[r, bend left, "f^t"] \arrow[r, phantom, "\perp"] & B^t \arrow[l, bend left, "u^t"]
\end{tikzcd}$$

- Now if $\mathbb{A} : \mathcal{A}dj \to \mathcal{K}$ is any other homotopy coherent adjunction with homotopy coherent monad $\mathbb{T}$, the universal property of the comparison map induces a simplicial natural transformation $\mathbb{A} \to \mathbb{A}^{\mathbb{T}}$, the non-identity component of which gives a canonical comparison map

$$\begin{tikzcd}
A \arrow[rr, dashed, bend left, "k"] \arrow[dr, bend right, "u"'] & & B^t \arrow[dl, bend right, "u^t"'] \\
& B \arrow[ul, bend right, "f"'] \arrow[ul, phantom, "\perp"] \arrow[ur, bend right, "f^t"'] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

⋆ The *monadicity theorem* gives conditions under which this map k is an equivalence. The *descent theorem* of Sulyma [Su] gives conditions under which k is fully faithful.

Wednesday, May 30

Using the simplicial cotensor with $\mathbb{2} = \Delta[1]$, for any ∞-category A we can form a span in $\mathcal{K}$ as below-left that is equipped with a 2-cell in the homotopy 2-category as below-right called the *generic arrow* over A:

$$\begin{tikzcd}
& A^{\mathbb{2}} \arrow[dl, "p_1 =: \mathrm{cod}"'] \arrow[dr, "p_0 =: \mathrm{dom}"] & \\
A & & A
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A^{\mathbb{2}} \arrow[d, bend right, "\mathrm{cod}"'] \arrow[d, bend left, "\mathrm{dom}"] \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] \\
A
\end{tikzcd}$$

This has a weak 2-categorical universal property that says that every other 2-cell over A as below left factors through the generic arrow as below right.

$$
\begin{tikzcd}
X \arrow[d, bend right, "y"'] \arrow[d, bend left, "x"] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}"] \\
A
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
X \arrow[d, "\ulcorner\alpha\urcorner"] \\
A^{\mathbb{2}} \arrow[d, bend right, two heads, "\mathrm{cod}"'] \arrow[d, bend left, two heads, "\mathrm{dom}"] \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] \\
A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[d, bend right, "y"'] \arrow[d, bend left, "x"] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}"] \\
A
\end{tikzcd}
$$

The span $A \xleftarrow{\mathrm{cod}} A^{\mathbb{2}} \xrightarrow{\mathrm{dom}} A$ has an additional property: there is a lifting condition of 2-cells along cod that can be summarized by the slogan that "arrows in the base A act covariantly on the fiber $A^{\mathbb{2}}$," where the action is given by post-composition:

$$
\begin{tikzcd}
X \arrow[r, "\ulcorner\alpha\urcorner"] \arrow[dr, bend left, "y"] \arrow[dr, bend right, "z"'] \arrow[dr, phantom, "\Downarrow\beta"] & A^{\mathbb{2}} \arrow[d, two heads, "\mathrm{cod}"] \\
& A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, bend left, "\ulcorner\alpha\urcorner"] \arrow[r, bend right, dashed, "\ulcorner\beta\circ\alpha\urcorner"'] \arrow[r, phantom, "\Downarrow\exists"] & A^{\mathbb{2}} \arrow[d, two heads, "\mathrm{cod}"] \\
& A
\end{tikzcd}
$$

This is summarized by saying that $\mathrm{cod} : A^{\mathbb{2}} \twoheadrightarrow A$ is a *cocartesian fibration*. Dually, $\mathrm{dom} : A^{\mathbb{2}} \twoheadrightarrow A$ is a *cartesian fibration*, with arrows in the base A acting contravariantly on the fiber $A^{\mathbb{2}}$ by precomposition. Moreover these "left" and "right" actions commute and there's an extra "discreteness" property of the combined map $(\mathrm{cod}, \mathrm{dom}) : A^{\mathbb{2}} \twoheadrightarrow A \times A$. The summary is that this span expresses $A^{\mathbb{2}}$ as a *module* from A to A.

9. Cartesian fibrations and the Yoneda lemma (Paul Lessard).

The theory of *cartesian* or *Grothendieck fibrations* play a fundamental role in traditional internal category theory. For example, they arise naturally in the theory of modules to be developed in the subsequent talk, which may be described as certain spans whose legs are cartesian fibrations and in the theory of *fibred* (or *indexed*) *categories* over a base.

Joyal and Lurie both introduce a cartesian fibration notion for quasi-categories, described in terms of certain outer horn lifting properties. We can generalise that notion *representably* to an arbitrary ∞-cosmos $\mathcal{K}$; that is to say we might adopt the definition that a fibration $p : E \to B$ in $\mathcal{K}$ is a cartesian fibration if and only if for all objects $X \in \mathcal{K}$ the functor $\mathsf{Fun}(X, p) : \mathsf{Fun}(X, E) \to \mathsf{Fun}(X, B)$ of hom-spaces is itself a cartesian fibration of quasi-categories and if any $Y \to X \in \mathcal{K}$ defines a cartesian functor between these cartesian fibrations.

As is our wont, however, we choose to take a more 2-categorical approach to this subject, by defining cartesian fibrations in terms of adjunctions and comma objects. We establish an equivalence between three characterisations of these structures defined internally to the homotopy 2-category. We also discuss *discrete* cartesian fibrations (called *right fibrations* by Joyal and Lurie), whose properties and applications mirror those of *discrete fibrations* in 1-category theory.

Now given a *point* $b: 1 \to B$ of an ∞-category, our categorical experience draws us to regarding the associated projection $p_B: B \downarrow b \to B$ as being the *representable* cartesian fibration defined by b. To validate that intuition, we discuss how the (external) Yoneda lemma may be formulated in this context and we prove that result.[2] This proof takes place entirely within the homotopy 2-category of our cosmos, as indeed do most of the proofs referenced in this talk.

References: [RV4, §4-6], [J], [L1, §2], [RV, chapter 5].

⋆ Fix an isofibration $p: E \twoheadrightarrow B$. A 2-cell $\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Uparrow\chi"] & E \end{tikzcd}$, which we can also write as $\chi : e' \Rightarrow e$, is *p-cartesian* if
- for any $\tau : e'' \Rightarrow e$ that has a factorization $p\tau = p\chi \cdot \gamma$ there exists a factorization $\tau = \chi \cdot \zeta$ with $p\zeta = \gamma$.
- if $\chi = \chi \cdot \epsilon$ and $p\epsilon = \mathrm{id}$, then ϵ is an isomorphism.

⋆ An isofibration $p: E \twoheadrightarrow B$ is a *cartesian fibration* if
- every 2-cell as below-left admits a lift as below-right that is p-cartesian

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, bend left=20, "pe"] \arrow[dr, bend right=30, "b"'] \arrow[dr, phantom, "\Uparrow\beta"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, dashed, "e'"'] \arrow[r, phantom, "\Uparrow\exists\chi"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd}$$

- the class of p-cartesian cells is stable under restriction along functors: if $\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Uparrow\chi"] & E \end{tikzcd}$ is p-cartesian so is $\begin{tikzcd} Y \arrow[r, "f"] & X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Uparrow\chi"] & E \end{tikzcd}$.

• There is an *internal characterization* of cartesian fibrations that is used in proving many of the facts about them involving a functor $k: E^2 \to B \downarrow p$ that we now define. The comma cone ϕ for $B \downarrow p$ displayed below right is the "generic arrow that should admit a p-cartesian lift." By 1-cell induction, there is a map $E^2 \twoheadrightarrow B \downarrow p$ that represents the composite $p\kappa$ where κ is the generic arrow for E:

$$\begin{tikzcd} E^2 \arrow[d, two heads, bend right, "p_1"'] \arrow[d, two heads, bend left, "p_0"] \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] \\ E \arrow[d, two heads, "p"] \\ B \end{tikzcd} \quad = \quad \begin{tikzcd} & E^2 \arrow[d, dashed, "k"] \arrow[ddl, two heads, bend right, "p_1"'] \arrow[ddr, two heads, bend left, "pp_0"] & \\ & B \downarrow p \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ E \arrow[rr, "p"'] & \; & B \end{tikzcd}$$

⋆ Now the map $p: E \twoheadrightarrow B$ is a cartesian fibration if and only if the functor $k: E^2 \to B \downarrow p$ admits a right adjoint for which the counit is an isomorphism. The idea is that right adjoint $\bar{r}: B \downarrow p \to E^2$

[2] An analogous analytic development of cartesian fibrations and the Yoneda lemma in the complete Segal space model can be found in [Ra1, Ra2].

represents a 2-cell $\begin{tikzcd} B \downarrow p \arrow[r, bend left, "\mathrm{cod}"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Uparrow\chi"] & E \end{tikzcd}$ that defines the p-cartesian lift of the 'generic arrow that should admit a p-cartesian lift" ϕ.

- There is a Yoneda lemma for cartesian fibrations. For any element $b : 1 \to B$, the domain-projection $\mathrm{dom} : B \downarrow b \twoheadrightarrow B$ is a cartesian fibration. The Yoneda lemma says that the quasi-category whose vertices are cartesian functors from $\mathrm{dom} : B \downarrow b \twoheadrightarrow B$ to a cartesian fibration $p : E \twoheadrightarrow B$ is equivalent to the quasi-category whose vertices are functors from $b : 1 \to B$ to $p : E \twoheadrightarrow B$ over B. The equivalence is implemented by restricting along the identity element $\ulcorner \mathrm{id}_b \urcorner : 1 \to B \downarrow b$.

10. Two-sided fibrations and modules (Daniel Fuentes-Keuthan).

Our goal in this talk, and its successor, is to provide a *modular* (or *profunctorial*) foundation for the category theory of ∞-categories in an ∞-cosmos $\mathcal{K}$. Specifically, we follow Street by developing a theory of *two-sided cartesian fibrations* over a pair of ∞-categories $A, B \in \mathcal{K}$. We might think of these as families of ∞-categories, indexed jointly by A and B, which possess compatible actions of A on the left (a covariant action) and of B on the right (a contravariant action). In Street's presentation of this notion, two-sided fibrations are represented as spans $q : A \twoheadleftarrow C \twoheadrightarrow B : p$ in which

- p is a cartesian fibration whose cartesian arrows map to isomorphisms under q,
- q is a cocartesian fibration whose cocartesian arrows map to isomorphisms under p, and
- (co-)cartesian lifts along p and q satisfy a *Beck-Chevalley condition.*

We take a slightly different approach, also originally suggested by Street, which realises two-sided fibrations as certain cartesian (cocartesian) fibrations in an ∞-cosmos of cocartesian (cartesian) fibrations.

We start by observing that if $\mathcal{K}$ is an ∞-cosmos then the simplicial category $\mathcal{C}art(\mathcal{K})_{/B}$ of cartesian fibrations and *cartesian functors* over a fixed base ∞-category $B \in \mathcal{K}$ is again an ∞-cosmos. Indeed, we can go a little further and describe a ∞-cosmos $\mathcal{C}art(\mathcal{K})$ of cartesian fibrations over arbitrary base ∞-categories and cartesian functors; this is also an ∞-cosmos and which admits a cosmological functor $\mathrm{cod} : \mathcal{C}art(\mathcal{K}) \to \mathcal{K}$ whose fibres are the $\mathcal{C}art(\mathcal{K})_{/B}$.

Now we are ready to define our two-sided fibrations between ∞-categories $A, B \in \mathcal{K}$ to be cocartesian fibrations over the projection $A \times B \twoheadrightarrow B$ in the ∞-cosmos $\mathcal{C}art(\mathcal{K})_{/B}$ of cartesian fibrations over B. We show that this definition unwinds to give a more symmetric one, in the spirit of Street, and that in the dual this is no-more-nor-less than a cartesian fibration over the projection $A \times B \twoheadrightarrow A$ in the ∞-cosmos $co\mathcal{C}art(\mathcal{K})_{/A}$ of cocartesian fibrations over A. Consequently, by iterating the ∞-cosmos of (co)cartesian fibrations construction, we obtain an ∞-cosmos of two-sided fibrations

$$_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \coloneqq co\mathcal{C}art(\mathcal{C}art(\mathcal{K})_{/B})_{/(\pi_B : A \times B \to B)} \cong \mathcal{C}art(co\mathcal{C}art(\mathcal{K})_{/A})_{/(\pi_A : A \times B \to A)}$$

over fixed ∞-categories A and B in $\mathcal{K}$. Furthermore, we define the simplicial category $_{A\backslash}\mathcal{M}od(\mathcal{K})_{/B}$ of *modules* from A to B to be the full simplicial sub-category of *discrete* two-sided fibrations in $_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B}$.

The utility of framing our definitions in this way lies in the fact that we may now apply any result that lifts structures or properties from an ∞-cosmos $\mathscr{K}$ to the associated ∞-cosmoi of (co)cartesian fibrations to provide a corresponding lifted entity for the ∞-cosmos ${}_{A\backslash}\mathscr{F}ib(\mathscr{K})_{/B}$.

References: [RV7, §5], [RV9], [RV5, §3], [RV, chapter 10].

- ⋆ A unital ring R is the same thing as a $\mathscr{A}b$-enriched category with one object. Note that "actual modules" — say a bimodule M from with a left action by a unital ring R and a right action by a unital ring S — can be thought of as "$\mathscr{A}b$-enriched profunctors," i.e.,: $R \otimes S^{\mathrm{op}} \to \mathscr{A}b$. This, and the fact that they will have the same formal properties, is why we use the name "modules" for the concept to be defined below.
- ⋆ A *module* will be a *two-sided fibration* $A \overset{q}{\leftarrow} E \overset{p}{\to} B$ (with both maps isofibrations) that is additionally *discrete*. We need two define both of these terms.
- A span of isofibrations $A \overset{q}{\leftarrow} E \overset{p}{\to} B$ is *cocartesian on the left* if q is a cocartesian fibration and if whenever χ is a q-cocartesian cell then $p\chi$ is invertible. There are various equivalent characterizations of this. Dually a span of isofibrations is *cartesian on the right* if p is a cartesian fibration and if whenever χ is a p-cartesian cell then $q\chi$ is invertible.
- Using the equivalent characterizations eluded to above, if the span of isofibrations $A \overset{q}{\leftarrow} E \overset{p}{\to} B$ is both cocartesian on the left and cartesian on the right this means that

$$\begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

is a diagram in the cosmos $co\mathscr{C}art(\mathscr{K})_{/A}$ and is a cartesian fibration in $\mathscr{K}_{/A}$. Dually, span of isofibrations $A \overset{q}{\leftarrow} E \overset{p}{\to} B$ is both cocartesian on the left and cartesian on the right if and only if

$$\begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

is a diagram in the cosmos $\mathscr{C}art(\mathscr{K})_{/B}$ and is a cocartesian fibration in $\mathscr{K}_{/A}$.
- ⋆ A span of isofibrations $A \overset{q}{\leftarrow} E \overset{p}{\to} B$ is a *two-sided fibration* just when the diagram below-left

$$\begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

is a cartesian fibration in $co\mathscr{C}art(\mathscr{K})_{/A}$ and the diagram above-right is a cocartesian fibration in $\mathscr{C}art(\mathscr{K})_{/B}$.
- ⋆ What this means is the following: given $e : X \to E$ and a 2-cell $\alpha : qe \Rightarrow a$ and a 2-cell $\beta : b \Rightarrow pe$, we get two new maps $\beta^*e, \alpha_*e : X \to E$ by taking p-cartesian and q-cartesian lifts. We can do this again to get two maps $\alpha_*\beta^*e, \beta^*\alpha_*e : X \to E$ and the conditions above say that these are isomorphic over $A \times B$.

- The ∞-*cosmos of two-sided fibrations* from A to B can be defined in two isomorphic ways:

$$_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} := co\mathcal{C}art(\mathcal{C}art(\mathcal{K})_{/B})_{/(\pi_B\colon A\times B\to B)} \cong \mathcal{C}art(co\mathcal{C}art(\mathcal{K})_{/A})_{/(\pi_A\colon A\times B\to A)}$$

  The functors in this ∞-cosmos are the maps of spans that simultaneously define a cartesian functor on the right side and a cocartesian functor on the left side. The ∞-cosmos structure is created by the inclusion $\mathcal{K}_{/A\times B}$.

★ Two-sided fibrations are stable under pullback. Since $A \xleftarrow{\text{cod}} A^2 \xrightarrow{\text{dom}} A$ is a two-sided fibration, so are all comma spans $C \xleftarrow{\text{cod}} f \downarrow g \xrightarrow{\text{dom}} B$.

- You can also take a two-sided fibration $A \overset{q}{\leftarrow} E \overset{p}{\rightarrow} B$ and compose with a cartesian fibration $B \twoheadrightarrow B'$ and a cocartesian fibration $A \twoheadrightarrow A'$ to get a two-sided fibration $A' \leftarrow E \rightarrow B'$.
- Combining these properties, the "horizontal composite" of two-sided fibrations is a two-sided fibration:

$$\begin{tikzcd}
 & & E \times_B F \arrow[dl, two heads, "\pi_1"'] \arrow[dr, two heads, "\pi_0"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] & & F \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "r"] & \\
A & & B & & C
\end{tikzcd}$$

- An object $E \in \mathcal{K}$ is *discrete* if every 2-cell with codomain E is invertible iff $\mathsf{Fun}(X, E)$ is a Kan complex (not merely a quasi-category). An object $p\colon E \twoheadrightarrow B \in \mathcal{K}_{/B}$ is discrete in the sense just defined if for any 2-cell α with codomain E, if $p\alpha$ is invertible then α is invertible. So $p\colon E \twoheadrightarrow B$ is discrete if and only if it is conservative.

★ A *module* is a two-sided fibration that is discrete as an object $E \twoheadrightarrow A\times B \in \mathcal{K}_{/A\times B}$. If $\mathcal{K} = q\mathcal{C}at$ this just means it's a two-sided fibration whose fibers are Kan complexes.

- Modules are pullback stable: if $(q, p)\colon E \twoheadrightarrow A \times B$ is a module then the pullback

$$\begin{tikzcd}
E(a,b) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "{(q,p)}"] \\
A' \times B' \arrow[r, "a\times b"'] & A \times B
\end{tikzcd}$$

  is a module from A' to B'. But the "horizontal composite" of modules as above is not discrete so is not a module.

★ The prototypical example of a module is $A \xleftarrow{\text{cod}} A^2 \xrightarrow{\text{dom}} A$. By pullback stability, comma spans are also modules. This will be very important.

- The horizontal composite of $A \xleftarrow{\text{cod}} A^2 \xrightarrow{\text{dom}} A$ with itself is $A \xleftarrow{\text{ev}_2} A^3 \xrightarrow{\text{ev}_0} A$ which is not a module.

Thursday, May 31

What is a module between ∞-categories and why do we care?

★ A module axiomatizes the properties of the bifunctor "Hom_A" for an ∞-category A as encoded by the span of isofibrations $A \xleftarrow{\text{cod}} A^2 \xrightarrow{\text{dom}} A$. Namely:

– The fibers over any pair of elements are *discrete* ∞-categories, which define the internal homs in A:

$$\begin{tikzcd} \mathrm{Hom}_A(x,y) \coloneqq x \downarrow y \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & A^{\mathbb{2}} \arrow[d, two heads, "{(\mathrm{cod},\mathrm{dom})}"] \\ 1 \arrow[r, "{(y,x)}"'] & A \times A \end{tikzcd}$$

– But the module $A^{\mathbb{2}}$ isn't just a family of internal hom-spaces $\mathrm{Hom}_A(x,y)$: in addition, arrows in A act covariantly in the codomain variable by postcomposition and contravariantly in the domain variably by precomposition, and these actions commute. These properties are expressed by saying that $(\mathrm{cod},\mathrm{dom}) : A^{\mathbb{2}} \twoheadrightarrow A \times A$ is cartesian on the right and cocartesian on the left.

• These properties are stable under pullback, so in particular for a functor $f : B \to A$ we obtain modules

$$\begin{tikzcd} A \downarrow f \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & A^{\mathbb{2}} \arrow[d, two heads, "{(\mathrm{cod},\mathrm{dom})}"] \\ B \times A \arrow[r, "f \times A"'] & A \times A \end{tikzcd} \qquad \begin{tikzcd} f \downarrow A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & A^{\mathbb{2}} \arrow[d, two heads, "{(\mathrm{cod},\mathrm{dom})}"] \\ A \times B \arrow[r, "A \times f"'] & A \times A \end{tikzcd}$$

which are *right represented* or *left represented* by the functor $f : B \to A$. Similarly, general comma spans $(\mathrm{cod},\mathrm{dom}) : f \downarrow g \twoheadrightarrow C \times B$ are modules.

Now why do you care. The main reason has to do with the following mantra:

> **Most (all?) of the category theory of ∞-categories can be encoded as an equivalence between modules.**

To illustrate:

⋆ A functor $u : A \to B$ admits a left adjoint if and only if the module $B \downarrow u$ is *represented on the left*, meaning there exists $f : B \to A$ so that $f \downarrow A \simeq B \downarrow u$ as modules, i.e., over $A \times B$.

⋆ A diagram $d : 1 \to A^J$ admits a limit if and only if the module $\Delta \downarrow d$ of cones over A is *represented on the right*, meaning there exists $\ell : 1 \to A$ so that $A \downarrow \ell \simeq \Delta \downarrow d$ as modules, i.e., over $A \times 1$.

11. The calculus of modules (Matthew Weatherley).

Having introduced a module notion, our challenge now is to organise these together into a structure that abstracts their role in formal category theory. To that end, we shall exploit a formal analogy between the categorical calculii of bimodules between commutative rings and that of modules $M : A \nrightarrow B$ (between categories), an intuition often given concrete realisation by assembling the categories, functors, and modules of an abstract category theory together into a structure dubbed a *pro-arrow equipment* (or simply just an *equipment*) by Wood.

Naïvely we might hope to assemble the ∞-categories, functors, and modules in an ∞-cosmos $\mathscr{K}$ into an equipment, but this is not an utopia universally available to us. Specifically, Wood's framework assumes that a pair of modules $M : A \nrightarrow B$ and $N : B \nrightarrow C$ may be composed to give a *tensor product* module $M \otimes N : A \nrightarrow C$ but our ∞-cosmos axiomatization is too sparse to construct tensor products in general. Consequently, we are forced to frame the labours of this talk within the more general theory of *virtual equipments* as presented by Cruttwell and Shulman.

Virtual equipments encapsulate a calculus whose primary protagonists are objects supporting two flavours of arrows, called *functors* and *modules*, and structures relating them called *cells* (which generalise natural transformations of modules). Cells may be depicted as rectangular tiles, whose vertical edges are functors and whose horizontal edges are (sequences of) modules, and they admit an associative vertical composition which acts to combine cells that abut along their horizontal edges. We shall display these composites as tiled regions called *pasting diagrams*, and very many of our arguments will be couched largely in diagrammatic terms.[3]

To motivate the claim that virtual equipments encapsulate a theory of modules suited to the expression of abstract category theory and to build some familiarity with this calculus, we re-derive some useful categorical results entirely within that formalism. We shall find that the functors $f : A \to B$ of any virtual equipment give rise to pairs of modules $B \downarrow f : A \nrightarrow B$ and $f \downarrow B : B \nrightarrow A$, that these are formally adjoint (in a suitable sense), that they admit certain (universally defined) tensor products with other modules, and that they satisfy various formulations of the Yoneda lemma. We also recover the module characterization of adjoint functors observed in talk 6.

We conclude this talk with a proof that the totality of ∞-categories, functors and modules in any ∞-cosmos $\mathscr{K}$ may indeed be collected together into a virtual equipment $\mathscr{M}od(\mathfrak{h}\mathscr{K})$ whose cells are given as certain transformations of modules.

References: [CS], [RV5, §4], [W], [RV, chapter 11]

- A *double category* has objects, horizontal arrows, vertical arrows, and cells that fit in squares. Horizontal and vertical arrows admit composites and squares can be composed in both horizontal and vertical directions.

⋆ A *virtual double category* is like a double category but where there no longer exist horizontal composites of horizontal arrows or squares. To make up for this, we allow the cells to fit in squares with an n-ary source of horizontal arrows for any $n \geq 0$:

$$\begin{tikzcd}
A_0 \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, "E_2", "\shortmid"{marking}] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & A_n \arrow[d, "g"] \\
B_0 \arrow[rrr, "F"', "\shortmid"{marking}] & & & B_n
\end{tikzcd}$$

including those whose horizontal source has length zero, in the case $A_0 = A_n$. These cells cannot be composed horizontally but can be composed vertically in a multi-categorical sort of way: for any configuration as below-left

$$\begin{tikzcd}
A_0 \arrow[r, "{E_{11},\ldots,E_{1k_1}}", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow"] & A_1 \arrow[r, "{E_{21},\ldots,E_{2k_2}}", "\shortmid"{marking}] \arrow[d, "f_1"] \arrow[dr, phantom, "\Downarrow"] & \cdots \arrow[r, "{E_{n1},\ldots,E_{nk_n}}", "\shortmid"{marking}] \arrow[d, "\cdots"] \arrow[dr, phantom, "\Downarrow"] & A_n \arrow[d, "f_n"] \\
B_0 \arrow[r, "F_1", "\shortmid"{marking}] \arrow[d, "g"'] & B_1 \arrow[r, "F_2", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & \cdots \arrow[r, "F_n", "\shortmid"{marking}] & B_n \arrow[d, "h"] \\
C_0 \arrow[rrr, "G"', "\shortmid"{marking}] & & & C_n
\end{tikzcd}
\quad =: \quad
\begin{tikzcd}
A_0 \arrow[r, "E_{11}", "\shortmid"{marking}] \arrow[d, "gf_0"'] & \cdot \arrow[r, "\cdots", "\shortmid"{marking}] & \cdots \arrow[r, "\cdots", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & \cdot \arrow[r, "E_{nk_n}", "\shortmid"{marking}] & A_n \arrow[d, "hf_n"] \\
C_0 \arrow[rrrr, "G"', "\shortmid"{marking}] & & & & C_n
\end{tikzcd}$$

there exists a composite cell as above-right.

---

[3] See [M] for a graphical calculus describing such diagrams.

⋆ A virtual double category is a *virtual equipment* if it satisfies two additional axioms:
  – The data below-left can be completed to a unary cell as below right

$$
\begin{tikzcd}
A' \arrow[d, "a"'] & B' \arrow[d, "b"] \\
A \arrow[r, "E"', "|" marking] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A' \arrow[r, "E(b,a)", "|" marking] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\rho"] & B' \arrow[d, "b"] \\
A \arrow[r, "E"', "|" marking] & B
\end{tikzcd}
$$

  that has a universal property that any n-arrow cell over the data below left factors uniquely through the unary cell as above right.
  – Any object A is equipped with a nullary cell

$$
\begin{tikzcd}
A \arrow[r, equal, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[d, equal] \\
A \arrow[r, "\mathrm{Hom}_A"', "|" marking] & A
\end{tikzcd}
$$

  that has the universal property that any cell which includes A as one of the objects in its horizontal source factors uniquely through this nullary cell.
⋆ The whole point of this is that any ∞-cosmos has a *virtual equipment of modules* $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ whose objects are the ∞-categories, whose vertical arrows are the functors, whose horizontal arrows are the modules, and whose cells are fibered isomorphism classes of maps of spans

$$
\begin{tikzcd}
A_0 \arrow[r, "E_1", "|" marking] \arrow[d, "f"'] & A_1 \arrow[r, "E_2", "|" marking] \arrow[d, phantom, "\Downarrow"] & \cdots \arrow[r, "E_n", "|" marking] & A_n \arrow[d, "g"] \\
B \arrow[rrr, "F"', "|" marking] & & & C
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
& E_1 \times_{A_1} \cdots \times_{A_{n-1}} E_n \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, dashed] & \\
A_0 \arrow[d, "f"'] & & A_n \arrow[d, "g"] \\
B & F \arrow[l, two heads] \arrow[r, two heads] & C
\end{tikzcd}
$$

• There are heaps of formal categorical properties that follow from these axioms, particularly involving the right-representable and left-representable modules $A \downarrow f : B \nrightarrow A$ and $f \downarrow A : A \nrightarrow B$ associated to $f : B \to A$.
⋆ In a virtual double category, unary cells can always be composed vertically. Importantly two parallel modules $E : A \nrightarrow B$ and $F : A \nrightarrow B$ are vertically isomorphic (via vertically-invertible cells whose vertical boundaries are identities) if and only if $E \simeq F$ over $A \times B$.
• The virtual equipment $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ contains the homotopy 2-category $\mathfrak{h}\mathcal{K}$ as the *vertical 2-category* comprised of only unary cells whose horizontal sources and targets are the horizontal units Hom_A, Hom_B, … .
• The homotopy 2-category also embeds both covariantly and contravariantly pseudofunctorially into the *horizontal "bicategory"* comprised of only unary cells whose vertical sources and targets are identity functors.
• In summary, the virtual equipment of modules contains the homotopy 2-category and naturally expresses equivalence of modules and their various properties.

12. Pointwise Kan extensions (Kevin Carlson).

The adjoint formulation of (co)limits in ∞-categories, as presented in talk 4, is adequate for many purposes but is found wanting when we come to consider the theory of *Kan extensions*. Any 2-category supports a notion of Kan extension, couched in terms of a universal property of 2-cell bearing triangles, and this may be imported via the homotopy 2-category into the theory of ∞-categories in an ∞-cosmos. It is known, however, that even in the 2-category of categories this does not characterise the class of Kan extensions of primary interest, that is those that may be regarded as being constructed *pointwise* from the (co)limits that may exist in the target category. What is more, our existing theory of (co)limits does not easily provide us with an analogue of Kan's formula for constructing these extensions. This talk rectifies these deficiencies.

We start by defining what it means for an ∞-category to admit a family of (co)limits weighted by a module, and we derive some basic consequences. This then leads to a theory of *pointwise Kan extensions* which may be applied in any virtual equipment. At this level of generality, we also introduce an *exact square* notion and we examine the sense in which this provides a common language in which to discuss the theory of functors that are fully faithful, final, or initial.

It is also possible to characterise pointwise Kan extensions in terms expressible entirely within the homotopy 2-category associated with an ∞-cosmos. We discuss this alternative characterisation and demonstrate that it is equivalent to the equipment based notion. Finally we specialise these notions to ∞-cosmoi that are cartesian closed, we prove familiar properties of initial and final functors and a Beck-Chevalley result for pointwise Kan extensions.

Specialising all of this to the ∞-cosmos of quasi-categories, we prove the expected Kan extension existence result for functors that land in suitable (co)complete quasi-categories. Ultimately this leads us to a proof that any suitably complete and cocomplete quasi-category gives rise to a *derivator* in the sense of Heller [He] and Grothendieck [G].

References: [RV5, §5], [RV9, §6], [RV, chapter 12].

- ⋆ Mac Lane famously claimed that "all concepts are Kan extensions." Our goal is to put this in the homotopy 2-category. You can define a right Kan extension in any 2-category but in $\mathfrak{h}\mathscr{K}$ it's too weak.
- To get a sense of this, suppose our ∞-cosmos $\mathscr{K}$ is *cartesian closed*, which means it has a simplicially enriched right adjoint to the product $A \times - : \mathscr{K} \to \mathscr{K}$. Then an absolute right lifting diagram, with A, B, C all ∞-categories as below right transposes to a diagram as on the left

$$\begin{tikzcd} & C^B \arrow[d, "k^*"] \\ 1 \arrow[ur, dashed, "r"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & C^A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[dr, dashed, "r"] \arrow[dr, phantom, shift right=3, "\Downarrow\rho"] & \\ A \arrow[u, "k"] \arrow[r, "f"'] & C \end{tikzcd}$$

that is a *right Kan extension* that is also stable under pasting with squares of the form below-left, meaning that the diagram below is still a right extension diagram.

$$\begin{tikzcd} B\times X \arrow[r, "\pi"] & B \arrow[dr, dashed, "r"] \arrow[d, phantom, "\Downarrow\rho"] & \\ A\times X \arrow[u, "k\times X"] \arrow[r, "\pi"'] & A \arrow[u, "k"] \arrow[r, "f"'] & C \end{tikzcd}$$

The point is just saying the triangle is a right Kan extension is not enough.

★ Kelly famously claimed that "all important concepts are pointwise Kan extensions." A *pointwise right Kan extension* is one that has a good reason to exist or, more precisely, one that can be computed by a limit formula.

★ In $\mathscr{C}at$, a right Kan extension

$$\begin{tikzcd} B \arrow[dr, dashed, "r"] \arrow[d, phantom, shift left=3ex, "\Downarrow\rho"] & \\ A \arrow[u, "k"] \arrow[r, "f"'] & C \end{tikzcd}$$

is pointwise just when for all $b\colon 1 \to B$, $rb \cong \lim(b{\downarrow}k \xrightarrow{p_1} A \xrightarrow{f} C)$. From the above transposition, this pointwise criterion in particular asserts that

$$\begin{tikzcd} 1 \arrow[r, "b"] \arrow[dr, phantom, "\Downarrow\phi"] & B \arrow[dr, dashed, "r"] \arrow[d, phantom, shift left=3ex, "\Downarrow\rho"] & \\ b\downarrow k \arrow[u, "p_0"] \arrow[r, "p_1"'] & A \arrow[u, "k"] \arrow[r, "f"'] & C \end{tikzcd}$$

- Define a *pointwise right extension* to be a right Kan extension diagram in $\mathfrak{h}\mathscr{K}$ so that for any comma square, the pasted composite is still a right Kan extension diagram in $\mathfrak{h}\mathscr{K}$.

$$\begin{tikzcd} B \arrow[dr, dashed, "r"] \arrow[d, phantom, shift left=3ex, "\Downarrow\rho"] & \\ A \arrow[u, "k"] \arrow[r, "f"'] & C \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r, dashed, "b"] \arrow[dr, phantom, "\Downarrow\phi"] & B \arrow[dr, dashed, "r"] \arrow[d, phantom, shift left=3ex, "\Downarrow\rho"] & \\ b\downarrow k \arrow[u, "p_0"] \arrow[r, "p_1"'] & A \arrow[u, "k"] \arrow[r, "f"'] & C \end{tikzcd}$$

- More generally, pointwise right extensions in $\mathfrak{h}\mathscr{K}$ are stable under pasting with all *exact squares*, which include lots of other examples.
- There is an equivalent characterization of this in the virtual equipment $\mathscr{M}od(\mathfrak{h}\mathscr{K})$: $\rho : rk \Rightarrow f$ is a pointwise right extension if and only if the binary cell

$$\begin{tikzcd} A \arrow[r, "B\downarrow k", "\vert" marking] \arrow[d, equal] & B \arrow[r, "C\downarrow r", "\vert" marking] \arrow[d, phantom, "\Downarrow\rho"] & C \arrow[d, equal] \\ A \arrow[rr, "C\downarrow f"', "\vert" marking] & & C \end{tikzcd}$$

is a right extension in $\mathscr{M}od(\mathfrak{h}\mathscr{K})$.

★ In a non cartesian closed ∞-cosmos, pointwise right and left Kan extensions can be used to define limits and colimits of functors $d : J \to A$ where J and A are both ∞-categories. The limit is just the pointwise right extension

$$\begin{tikzcd} 1 \arrow[dr, dashed, "\lim"] & \\ J \arrow[u, "k"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\epsilon"] & C \end{tikzcd}$$

13. Proof of model-independence (Jonathan Weinberger).

The biequivalences of ∞-cosmoi introduced in talk 3 might be referred to as "change-of-model" functors, converting complete Segal spaces to Segal categories or quasi-categories for instance. In this talk we prove that a biequivalence of ∞-cosmoi induces a biequivalence between the corresponding calculii of modules as expressed by the virtual equipment of ∞-categories, functors, modules, and module maps. The corollary is that categorical results proven with any of the biequivalent models of $(\infty, 1)$-categories apply to them all.

In more detail, a biequivalence of ∞-cosmoi $\mathscr{K} \xrightarrow{\sim} \mathscr{L}$ induces a biequivalence between their homotopy 2-categories — this being a 2-functor that is essentially surjective on objects up to equivalence and defines a local equivalence of functor spaces — that in addition preserves and indeed reflects and creates the comma ∞-category of any cospan. As a corollary, the biequivalence induces a bijection between equivalence classes of objects, a local bijection between isomorphism classes of parallel functors, a local bijection between natural transformations, a local bijection between equivalence classes of modules, and so on. It follows that a functor in $\mathscr{K}$ admits a right adjoint if and only if its image does so in $\mathscr{L}$ or a diagram valued in an ∞-category in $\mathscr{K}$ admits a limit if and only if any diagram in an ∞-category in $\mathscr{L}$ that is equivalent to its image admits a limit there. Since "formal category theory" enables us to rephrase categorical statements in terms of equivalences between modules, we conclude more generally that the formal category theory of ∞-categories in $\mathscr{K}$ is equivalent to the formal category theory of ∞-categories in $\mathscr{L}$.

After establishing model independence, we embark upon a guided tour through applications of the speaker's choosing, illustrating how a change-of-model functor can be used to transfer an "analytically-proven" result about one model of $(\infty, 1)$-categories to another model. Sample applications of this kind can be found in [RV10] but we encourage the speaker to search for their own.

References: [RVx], [RV10], [RV, part IV].

- ⋆ Formal ∞-category encompasses equivalences, adjunctions, limits and colimits, and cartesian fibrations, among other topics.
- ⋆ The informal statement of model-independence is that:

  Equivalent ∞-cosmoi have equivalent ∞-category theories.
- ⋆ A *cosmological functor* $F : \mathscr{K} \to \mathscr{L}$ is a simplicial functor that preserves isofibrations and the distinguished simplicially enriched limits.
- • Examples include $\mathscr{C}at \hookrightarrow q\mathscr{C}at$, $\mathsf{Fun}(A, -) : \mathscr{K} \to q\mathscr{C}at$, $(-)^2 : \mathscr{K} \to \mathscr{K}$ or any other simplicial cotensor, $f^* : \mathscr{K}_{/B} \to \mathscr{K}_{/A}$ for any $f : A \to B$, and $F : \mathscr{K}_{/B} \to \mathscr{L}_{/FB}$ for any cosmological functor $F : \mathscr{K} \to \mathscr{L}$.
- • In particular, any ∞-cosmos $\mathscr{K}$ has an *underlying quasi-category functor*

$$(-)_0 \coloneqq \mathsf{Fun}(1, -) : \mathscr{K} \to q\mathscr{C}at.$$

- There are many other examples as well involving ∞-cosmoi that we haven't discussed.

★ Why do we care about cosmological functors? They have a very useful property:

> **Cosmological functors preserve all ∞-categorical notions that can be characterized internally to the ∞-cosmos.**

This includes:
  - equivalences between ∞-categories and trivial fibrations
  - discrete ∞-categories — those A so that $A^{\mathbb{I}} \twoheadrightarrow A^{2}$ is a trivial fibration
  - comma ∞-categories, which are constructed by a simplicial pullback (or equivalent to such a thing)
  - adjunctions
  - invertibility of 2-cells and mates
  - homotopy coherent adjunctions and monads
  - absolute lifting diagrams — because these can be characterized by a fibered equivalence
  - colimits or limits
  - cartesian or cocartesian fibrations and cartesian functors between them
  - discrete cartesian or cocartesian fibrations
  - two-sided fibrations and modules
  - representable modules

★ A *cosmological biequivalence* is a cosmological functor $F : \mathscr{K} \to \mathscr{L}$ that is additionally
  - essentially surjective on objects up to equivalence: for all $B \in \mathscr{L}$ there exists $A \in \mathscr{K}$ so that $FA \simeq B$
  - locally fully faithful: the map $\mathsf{Fun}_{\mathscr{K}}(A, B) \xrightarrow{\sim} \mathsf{Fun}_{\mathscr{L}}(FA, FB)$ is an equivalence of quasi-categories

Biequivalences do have an inverse in some sense but it's not a cosmological functor nor even strictly simplicially enriched.

- The following functors between models of $(\infty, 1)$-categories are biequivalences:

$$\begin{tikzcd}
\mathscr{C}\mathscr{S}\mathscr{S} \arrow[rr, "\text{discretization}"] \arrow[dr, shift left, "\mathrm{row}_0"] & & \mathscr{S}egal \arrow[dl, shift right, "\mathrm{row}_0"'] \\
& q\mathscr{C}at \arrow[ul, shift left, "\text{cylinder}"] \arrow[ur, shift right, "\text{prism}"'] \arrow[d, shift left, "\natural"] & \\
& 1\text{-}\mathscr{C}omp \arrow[u, shift left, "U"] &
\end{tikzcd}$$

- If $\mathscr{K}$ is biequivalent to $q\mathscr{C}at$ (perhaps via a finite zig-zag of biequivalences), then $(-)_0 : \mathscr{K} \to q\mathscr{C}at$ is a biequivalence.
- If $f : A \xrightarrow{\sim} B$ is an equivalence, then $f^* : \mathscr{K}_{/B} \xrightarrow{\sim} \mathscr{K}_{/A}$ is a biequivalence. If $F : \mathscr{K} \xrightarrow{\sim} \mathscr{L}$ is a biequivalence, then $F : \mathscr{K}_{/B} \xrightarrow{\sim} \mathscr{L}_{/FB}$ is a biequivalence.

★ If $F : \mathscr{K} \xrightarrow{\sim} \mathscr{L}$ is a biequivalence then the 2-functor $F : \mathfrak{h}\mathscr{K} \xrightarrow{\sim} \mathfrak{h}\mathscr{L}$ is a biequivalence: essentially surjective up to equivalence and a local equivalence of hom-categories.

★ Why do we care about cosmological biequivalences? They have a very useful property:

> **Cosmological biequivalences preserve, reflect, and create all ∞-categorical notions that can be characterized internally to the ∞-cosmos.**

These include all of the notions mentioned above and also exact squares and pointwise Kan extensions.

- There are various ways to embed the homotopy 2-category into the virtual equipment, including the "horizontal covariant embedding" which sends $f : A \to B$ in $\mathfrak{h}\mathscr{K}$ to $B \downarrow f : A \nrightarrow B$ in $\mathscr{M}od(\mathfrak{h}\mathscr{K})$.

⋆ The upshot of all of this is that a biequivalence $F : \mathscr{K} \xrightarrow{\sim} \mathscr{L}$ of ∞-cosmoi induces a *biequivalence of virtual equipments* $F : \mathscr{M}od(\mathfrak{h}\mathscr{K}) \xrightarrow{\sim} \mathscr{M}od(\mathfrak{h}\mathscr{L})$. Since all of ∞-category theory can be expressed in here, this proves that biequivalent ∞-cosmoi have equivalent ∞-category theories.

- As a suggested exercise: apply the above results to prove that in the presence of a biequivalence $\mathscr{K} \xrightarrow{\sim} \mathscr{L}$ a functor in one cosmos admits a left adjoint there iff an equivalent functor does in the other cosmos. Similarly show that a diagram in one cosmos has a limit in there iff an equivalent diagram in the other cosmos does.

Friday, June 1

The microcosm principle says that if you look at categories of some structure the category then has structures that resemble a categorification of that structure. For instance, the category of monoids is a monoidal category. Also the 2-category of monoidal categories is a monoidal 2-category. Similarly there is a cosmological functor $\mathrm{cod} : co\mathscr{C}art(\mathscr{K}) \to \mathscr{K}$ that has the property of being a 2-cartesian fibration and this is the macro view of where the comprehension construction comes from.

14. Comprehension and the Yoneda embedding (Liang Ze Wong).

Given a cocartesian fibration $p : E \twoheadrightarrow B$ between ∞-categories and an ∞-category A, the *comprehension construction* defines a homotopy coherent diagram that we call the *comprehension functor* indexed by the quasi-category $\mathsf{Fun}(A, B)$ of functors from A to B and valued in the $(\infty, 1)$-categorical core of the ∞-cosmos $co\mathscr{C}art(\mathscr{K})_{/B}$ of cocartesian fibrations over B. In the case $A = 1$, the comprehension functor defines a "straightening" of the cocartesian fibration. In the case where the cocartesian fibration is the universal one over the quasi-category of small ∞-categories, the comprehension functor converts a homotopy coherent diagram of shape A into its "unstraightening," a cocartesian fibration over A.

To explain the name, there is an analogy first observed by Street between the comprehension construction in set theory and Grothendieck's construction of the category of elements of a functor $F : C \to \mathscr{S}et$ as the category formed by pulling back the cocartesian fibration $\mathscr{S}et_* \to \mathscr{S}et$. In the ∞-categorical context, the Grothendieck construction is christened "unstraightening" by Lurie. In this context, its inverse, the "straightening" of a cocartesian fibration into a homotopy coherent diagram is particularly important, because such functors are intrinsically tricky to specify, in practice requiring an infinite hierarchy of homotopy coherent data.

The fact that the comprehension construction can be applied in any ∞-cosmos has an immediate benefit. The codomain projection functor $\mathrm{cod} : A^2 \to A$ defines a cocartesian fibration in the slice ∞-cosmos $\mathscr{K}_{/A}$, in which case the comprehension functor specializes to define the Yoneda embedding, a map from the *underlying quasi-category* $\mathsf{Fun}(1, A)$ of A into the quasi-category $\wp(A)$ of discrete cartesian fibrations over A. This homotopy coherent diagram carries an element $a : 1 \to A$

to dom : $A \downarrow a \to A$, a module from 1 to A. A direct analysis of this construction proves that the Yoneda embedding is fully faithful.

References: [RV6], [L1], [KV].

- ⋆ In $\mathscr{C}at$, a cartesian fibration $p\colon E \twoheadrightarrow B$ encodes a pseudofunctor $B^{\mathrm{op}} \to \mathscr{C}at$ defined on $b\colon 1 \to B$ by taking the fiber E_b of p. The axioms of a cartesian fibration mean you also get the action on arrows. If you have a cocartesian fibration then you get a pseudofunctor $B \to \mathscr{C}at$. We'll talk about these so we don't have to have the "op."
- ⋆ The main theorem is that if $p : E \twoheadrightarrow B$ is a cocartesian fibration in an ∞-cosmos $\mathscr{K}$ and A is another object in $\mathscr{K}$, then you get a homotopy coherent functor, called the *comprehension functor*, whose domain is a cofibrant simplicial category called the homotopy coherent realization of the functor space $\mathsf{Fun}(A, B)$ and whose target is $co\mathscr{C}art(\mathscr{K})_{/A}$ that is defined on objects $b \in \mathsf{Fun}(A, B)$ by forming the pullback

$$\begin{tikzcd} E_b \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "b"] & B \end{tikzcd}$$

The data of this homotopy coherent diagram can be encoded by a functor of quasi-categories

$$c_{p,A} : \mathsf{Fun}(A, B) \to \mathsf{CoCart}(\mathsf{K})_{/A}$$

valued in the quasi-category underlying the ∞-cosmos. This is defined by taking the Kan complex core of the functor quasi-categories and then applying the homotopy coherent nerve.
- ⋆ The point of doing this at this level of generality is it allows us to unify many examples:
  - In the case $A = 1$, this is a functor

$$c_p : B_0 = \mathsf{Fun}(1, B) \to \mathsf{K}$$

    valued in the quasi-category associated to the ∞-cosmos $\mathscr{K}$.
  - In the case $A = 1$, and $\mathscr{K} = q\mathscr{C}at$ this is a functor

$$c_p : B \to \mathsf{QCat}$$

    valued in the quasi-category of quasi-categories. This is the *straightening construction*, which turns a cocartesian fibration $p : E \twoheadrightarrow B$ into a homotopy coherent diagram $c_p : B \to \mathsf{QCat}$
  - The *unstraightening construction* in $q\mathscr{C}at$ is the case where p is the cocartesian fibration $\mathsf{qcat}_* \twoheadrightarrow \mathsf{qcat}$ where the total space is the quasi-category of small quasi-categories with a basepoint and the functor is the forgetful one. The comprehension functor is then

$$\mathsf{Fun}(B, \mathsf{qcat}) \to \mathsf{CoCart}(\mathsf{qCat})_{/B}.$$

  - ⋆ The *Yoneda embedding* is defined by applying this construction in the ∞-cosmos $\mathscr{C}art(\mathscr{K})_{/A}$. The cocartesian fibration is the one given by the "cocartesian on the left" structure of the module A^2, namely:

$$\begin{tikzcd} A^2 \arrow[rr, two heads, "(p_1,p_0)"] \arrow[dr, two heads, "p_0"'] & & A \times A \arrow[dl, "\pi"] \\ & A & \end{tikzcd}$$

The comprehension functor takes the form on the right below and the restriction

$$\begin{array}{ccccc} A_0 = \mathsf{Fun}_{\mathscr{K}}(1, A) & \longrightarrow & \mathsf{Fun}_{\mathscr{K}_{/A}}(A \xrightarrow{\text{id}} A, A \times A \xrightarrow{\pi} A) & \xrightarrow{c} & \mathsf{Cart}(\mathsf{K})_{/A} \\ 1 \xrightarrow{a} A & \longmapsto & A \xrightarrow{a \times A} A \times A & \longmapsto & A \downarrow a \xrightarrow{p_0} A \end{array}$$

defines the Yoneda embedding.

- The Yoneda embedding is fully faithful. As a corollary, every quasi-category is equivalent to the homotopy coherent nerve of a Kan-complex enriched category (namely the Kan-complex enriched core of $\mathscr{C}art(q\mathscr{C}at)_{/A}$ spanned by the representables $p_0 : A \downarrow a \twoheadrightarrow A$.
- See the lecture notes for a beautiful sketch of the proof by Ze.

15. On the construction of limits and colimits (Tim Campion).

To this point we've talked in great generality about the meta-theory of limits and colimits in ∞-categories, but we have not as yet demonstrated the completeness or cocompleteness of any specific ∞-category. In this talk we rectify this oversight by discussing how limits and colimits arise in the homotopy coherent nerves of Kan complex enriched categories. While most of the (co-)completeness results we discuss in this talk apply specifically to the ∞-cosmos of quasi-categories, they may be transported along the change of model functors discussed in talk 13 to provide analogous results in the biequivalent models of $(\infty, 1)$-categories.

We start by considering the theory of categories enriched in Kan complexes and defining what it means for these to admit a flexibly weighted *homotopy (co-)limit*. We observe, in particular, that the full sub-category of fibrant and cofibrant objects in any simplicially enriched model category, in the sense of Quillen, admits all (small) flexibly weighted homotopy (co-)limits, thus providing us with a substantial stock of examples.

Given a simplicial set X we consider its *homotopy coherent realisation* $\mathfrak{C}(X)$ and diagrams $D : \mathfrak{C}(X) \to \mathscr{A}$ of that shape in a Kan complex enriched category $\mathscr{A}$. We define an associated flexible weight W_p on $\mathfrak{C}(X)$ and show that homotopy limits in $\mathscr{A}$ of diagrams $D : \mathfrak{C}(X) \to \mathscr{A}$ and weighted by W_p actually provide a limit for the dual diagram $\hat{D} : X \to \mathfrak{N}(\mathscr{A})$ in the homotopy coherent nerve $\mathfrak{N}(\mathscr{A})$, a (typically large) quasi-category.

Applying these results to the Kan complex enriched category $\mathscr{C}art^g(q\mathscr{C}at)_{/B}$, of discrete fibrations over a fixed quasi-category B, we show that its homotopy coherent nerve $\wp(B)$ is small complete and cocomplete. Now an entirely formal argument, expressed in the virtual equipment of modules in $q\mathscr{C}at$, establishes that the Yoneda embedding $\mathscr{Y} : B \to \wp(B)$, as introduced in the last talk, preserves any limits that happen to exist in B. Now the observation that products, pullbacks of isofibrations, and limits of countable chains of isofibrations are all homotopy limits in $\mathscr{C}art^g(q\mathscr{C}at)_{/B}$ allows us to demonstrate that a quasi-category admits all small limits if it possesses all pullbacks, products, and limits of countable chains.

If time permits we shall extend these ideas to a study of homotopy colimits in an ∞-cosmos $\mathscr{K}$, and discuss conditions under which these may be lifted to the ∞-cosmos $\mathscr{C}art(\mathscr{K})_{/B}$. We also show that pullback along any functor $f : A \to B$ between ∞-categories in $\mathscr{K}$ gives rise to an ∞-cosmos functor $f^* : \mathscr{C}art(\mathscr{K})_{/B} \to$

$\mathscr{C}art(\mathscr{K})_{/A}$ which preserves any homotopy colimits that lift to those ∞-cosmoi in this manner. This result extends, under suitable conditions, to the ∞-cosmos ${}_{A\backslash}\mathscr{F}ib(\mathscr{K})_{/B}$ and we show that in the case $\mathscr{K} = q\mathscr{C}at$ this leads to a construction which provides us with a reflection from two-sided fibrations to modules.

References: [RV7], [RV9].

- ⋆ Let $\mathscr{C}$ be a Kan-complex enriched category and let J be a simplicial set. A homotopy coherent diagram $D : \mathfrak{C}[J] \to \mathscr{C}$ has a homotopy limit or homotopy colimit if and only if the adjoint transpose $D : J \to N\mathscr{C}$ has a limit of colimit in the quasi-categorical sense.
- To explain, in the Berger model structure on simplicial categories the fibrant objects (good for mapping into) are the Kan-complex enriched categories. The cofibrant objects (good for mapping out of) are the *simplicial computads* which are "built by attaching cells": as cell complexes from $\emptyset \hookrightarrow \mathbb{1}$ and maps $\Sigma[\partial\Delta[n]] \hookrightarrow \Sigma[\Delta[n]]$ where $\Sigma[X]$ is the simplicial category with two objects and one non-trivial hom-space which is X. The simplicial computads can also be characterized as those simplicial objects $A_\bullet$ in $\mathscr{C}at$ (with all the objects the same) so that each category A_n is free on some graph of *atomic* arrows and if $f \in A_n$ is atomic then its degenerate images $f \cdot \sigma^i \in A_{n+1}$ are also atomic.
- For any 1-category C you can build a simplicial computad $FU_\bullet C$ as a simplicial object with $FU_n C := (FU)^{n+1}C$ where $F \dashv U$ is the free-forgetful adjunction between $\mathscr{C}at$ and reflexive directed graphs. In particular, applying this construction to the ordinal categories defines the *homotopy coherent simplices*, which assemble into a functor $\mathbf{\Delta} \to s\mathscr{C}at$. For formal reasons this cosimplicial object gives an adjunction

$$\begin{tikzcd} s\mathscr{S}et \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & s\mathscr{C}at \arrow[l, bend left, "\mathfrak{N}"] \end{tikzcd}$$

  between the *homotopy coherent realization* $\mathfrak{C}$ and *homotopy coherent nerve* $\mathfrak{N}$.
- A *flexible weight* $W : \mathscr{J} \to s\mathscr{S}et$ is a projective cofibrant weight, meaning built from $\partial\Delta[n] \times \mathscr{J}(j, -) \hookrightarrow \Delta[n] \times \mathscr{J}(j, -)$.
- A *W-weighted homotopy limit* in a Kan complex enriched category $\mathscr{C}$ is an object $L \in \mathscr{C}$ together with a cone $\lambda : W \Rightarrow \mathscr{C}(L, F-)$ with the property that when you take any *flexible resolution* $\bar{W} \to W$ (meaning $\bar{W}$ is flexible and the map is a pointwise weak homotopy equivalence), the W-shaped cone defines is a natural equivalence of Kan complexes

$$\mathscr{C}(C, L) \xrightarrow{\sim} \{\bar{W}, \mathscr{C}(C, F-)\}_{\mathscr{J}}.$$

  This notion is equivalence invariant and independent of the flexible resolution. A *homotopy limit* is a 1-weighted homotopy limit.
- The canonical weight on $\mathfrak{C}[J]$ is the simplicial functor $W_J : \mathfrak{C}[J] \to s\mathscr{S}et$ defined by $W_J(j) = \mathrm{Hom}_{\mathfrak{C}[\Delta[0]\star J]}(\top, j)$ from the cone point to j. Then $W_J \to 1$ is a flexible resolution.
- The main theorem is that for A an ∞-category, A is complete if and only if A has products and pullbacks. There is a relative version too which says that when B is complete, a functor $f : A \to B$ preserves limits if and only if it preserves products and pullbacks.
- To prove this we make use of another fact that we also prove namely that the quasi-category underlying the ∞-cosmos of discrete cartesian fibrations over A has all limits.
- To prove all these facts in the 1-categorical setting, you:
  - First prove that $\mathscr{S}et^{A^{\mathrm{op}}}$ has all limits (by constructing them).

- Then prove that $\mathscr{Y} : A \hookrightarrow \mathcal{S}et^{A^{\mathrm{op}}}$ preserves limits and is fully faithful and so reflects limits.
- Finally you can prove that if A has products and pullbacks then it has all limits by proving that the construction you think should work has the correct universal property (gives the corresponding limit after applying the Yoneda embedding).

- The proof for ∞-categories proceeds along the same lines using the Yoneda embedding defined in the previous talk.
- One strategy, outlined in the talk, is to prove this first for quasi-categories and then derive the general theory for any ∞-cosmos using the fact that $A \in \mathscr{K}$ has all limits of shape J if and only if the quasi-category $\mathsf{Fun}(X, A)$ has all J-shaped limits for every X and these are preserved by the precomposition functors with any $f : Y \to X$.
- The paper actually does the general case first.

16. Other approaches to model-independent ∞-category theory (Nima Rasekh).

The synthetic theory of ∞-categories developed internally to an ∞-cosmos and its homotopy 2-category provides one approach to developing the "model-independent" theory of ∞-categories. Furthermore, the model independence theorem discussed in talk 12 proves that even "analytic" theorems, proven in a particular model of ∞-categories, can be transferred across any biequivalence of ∞-cosmoi to demonstrate that result in other equivalent models. In this way, we conclude that the theories of $(\infty, 1)$-categories presented in the quasi-category, complete Segal space, Segal category, or naturally marked quasi-category models are all the same.

But this is not the only approach to model independent ∞-category theory. Even in the case of $(\infty, 1)$-categories, the ∞-cosmos axioms were deliberately chosen to exclude certain models[4] out of a desire to simply the proofs in the development of ∞-category theory within an ∞-cosmos. In this concluding talk, we invite the speaker to sketch other approaches to model independent ∞-category theory chosen at his or her discretion and the group as a whole to comment on their relative advantages and disadvantages.

References: [L0], [To], [BS-P], [AFR], [C], …

- In 1967, Quillen developed *model categories* to try to understand the homotopy theory of commutative rings (which is hard because they're not an abelian category). Along the way he studied *simplicial categories*.
- In 1972, Segal introduced *Gamma spaces* to study infinite loop spaces.
- In 1973, Boardman and Vogt were studying operads in spaces and discovered the first *quasi-categories* (and gave the definition).
- In 1980, Dwyer and Kan introduced *relative categories* as part of their work on simplicial localization.
- In 1989, Dwyer, Kan and Smith defined *Segal categories* motivated by the Γ-space construction of Segal.
- In 2001, Rezk introduced *complete Segal spaces* because he wanted a simplicial model category of $(\infty, 1)$-categories and none of the previous models had this property.

[4] Simplicially enriched categories or relative categories, both strictly-defined categorical objects that nonetheless define the objects in a model category that is Quillen equivalent to the other models, do not have well-behaved function complexes and hence do not fit into the ∞-cosmological framework.

- In this way, we arrived at six models of $(\infty, 1)$-categories: two strict models — simplicial categories and relative categories — and three weak models — quasi-categories, Segal categories, and complete Segal spaces. It's natural to ask how these are related:
  - How are we justified in thinking of these models as "the same"?
  - Is there a larger axiomatic structure that encompasses these?
- Model category structures were introduced on simplicial categories (Bergner '07), on quasi-categories (Joyal '90s), on relative categories (Barwick and Kan '12), on Segal categories (Hirschowitz & Simpson '01), and on complete Segal spaces (Rezk '01).
- So a way to make the second question more precise is to ask for an axiomatic framework on a model category that implies that it's objects are a model of $(\infty, 1)$-categories.
- This is done by Töen. The key feature required of the model category $\mathscr{M}$ is that it is equipped with a functor $C : \mathbf{\Delta} \to \mathscr{M}$ where the idea is that $C(0)$ is the free point, $C(1)$ is the free arrow, $C(2)$ is the free composable pair, and so on. If $\mathscr{M}$ is replaced by a Quillen equivalent simplicial model category, then this cosimplicial object can be used to turn any $X \in \mathscr{M}$ into a simplicial space $\mathsf{Map}(C(\bullet), X)$.
- Töen then gives seven axioms on the pair $(\mathscr{M}, C)$ and proves that any model category satisfying these axioms is Quillen equivalent to the complete Segal spaces model structure via $\mathsf{Map}(C(\bullet), -)$. He also proves that the space of automorphisms of a theory of $(\infty, 1)$-categories is $\mathbb{Z}/2$.
- A quasi-category A is *presentable* if there is a small quasi-category C and a fully faithful right adjoint

$$\begin{tikzcd} \mathsf{Fun}(C^{\mathrm{op}}, \mathsf{Spaces}) \arrow[r, bend left, "\ell"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "i"] \end{tikzcd}$$

  This embedding gives a "presentation" of A as simplicial sheaves. For example the quasi-category CSS has a very natural presentation in $\mathsf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Spaces})$. A modern viewpoint of Töen's result is that the axiomatization characterizes the quasi-category of $(\infty, 1)$-categories.
- Töen's result gives an equivalence of $(\infty, 1)$-categories of $(\infty, 1)$-categories rather than an equivalence of $(\infty, 2)$-categories of $(\infty, 1)$-categories. But since complete Segal spaces are cartesian closed all of the other models are too in a weak sense and the existence of exponentials gives a way to see that this implies that the $(\infty, 2)$-categories are equivalent in a weak sense.
- In summary, the "pros" of this approach is that it gives an axiomatization of $(\infty, 1)$-categories that's now commonly accepted and is relatively low tech (if you assume a lot of hard categorical homotopy theory is "low tech"). A "con" is that it doesn't really give you a way to think about category theory in a model-independent way. For instance, there are notions of *left fibration* (i.e., a discrete cocartesian fibration) in $q\mathscr{C}at$ and in $\mathscr{CSS}$ but if you stare at them it's not obvious they coincide. (This can be proven by using a Quillen equivalence or something like that, but you don't want to build a Quillen equivalence for every single categorical concept you want to compare.)

17. Future vistas (Emily Riehl & Dominic Verity).

The mentors will outline work in progress and survey open problems.

This week we've talked about proving theorems *about* ∞-categories? But how would you go about proving a theorem *with* ∞-categories? We'll describe a possible procedure in four steps (one of which is optional). To illustrate, consider a conjecture of the following form:

"A particular functor $f : A \to B$ of ∞-categories has a limit."

Step 1: define the ∞-categories A and B and the ∞-functor in any models whatsoever, perhaps in three different models. If the model isn't an ∞-cosmos, move the resulting $f : A \to B$ to one.

Step 2: Pick any of the equivalent definitions of the ∞-categorical notion you're trying to prove. Eg, one definition of a limit is given by:

Definition ([RV, chapter 12])**.** *A functor $f : A \to B$ between ∞-categories has a limit if and only if there exists a pointwise right Kan extension:*

$$\begin{tikzcd} & 1 \arrow[dr, dashed, "\ell"] & \\ A \arrow[ur, "!"] \arrow[rr, "f"'] & \arrow[u, phantom, "\Downarrow\lambda"] & B \end{tikzcd}$$

Or another definition of a limit is given by:

Definition ([RV, chapter 2])**.** *In a cartesian closed ∞-cosmos, a functor $f : A \to B$ has a limit if and only if there exists an absolute right lifting*

$$\begin{tikzcd} & B \arrow[d, "\Delta"] \\ 1 \arrow[ur, dashed, "\ell"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=2, "\Downarrow\lambda"] & B^A \end{tikzcd}$$

Proposition ([RV, chapter 12])**.** *In a cartesian closed ∞-cosmos the previous two definitions are equivalent.*

If our ∞-categories are $(\infty, 1)$-categories, the following result is relevant:

Proposition ([RV, chapter 13])**.**

(1) Any ∞-cosmos that is biequivalent to a cartesian closed ∞-cosmos has exponential objects B^A with a natural equivalence

$$\mathsf{Fun}(X, B^A) \simeq \mathsf{Fun}(X \times A, B)$$

and is thus weakly cartesian closed.

(2) In any ∞-cosmos that is biequivalent to $q\mathscr{C}at$ we have for all ∞-categories A and B that

$$B^A \simeq B^{A_0},$$

that is, the exponential is equivalent to the simplicial cotensor with the underlying quasi-category of A.

Consequently, in an ∞-cosmos that is biequivalent to $q\mathscr{C}at$, the previous definition is equivalent to the following one:

Definition ([RV, chapter 2])**.** *In an ∞-cosmos biequivalent to $q\mathscr{C}at$, a functor $f : A \to B$ has a limit if and only if there exists an absolute right lifting*

$$\begin{tikzcd} & B \arrow[d, "\Delta"] \\ 1 \arrow[ur, dashed, "\ell"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=2, "\Downarrow\lambda"] & B^{A_0} \end{tikzcd}$$

of the equivalent functor $f : 1 \to B^A \simeq B^{A_0}$.

In particular, in this setting the theorem about decomposing simplicial-set indexed diagrams into products and pullbacks applies.

In a weakly cartesian closed ∞-cosmos, we can also form the ∞-category $\Delta \downarrow f$ of cones over f as the comma object associated to the cospan in the second definition (or the third if it applies). This defines a module from 1 to B.

Definition ([RV, chapter 4]). *In a weakly cartesian closed ∞-cosmos, $f : A \to B$ has a limit $\ell : 1 \to B$ if and only if $\Delta \downarrow f \simeq B \downarrow \ell$ over B, that is, if and only if the module $\Delta \downarrow f$ is represented on the right by ℓ.*

Proposition ([RV, chapter 4]). *In a weakly cartesian closed ∞-cosmos, this definition is equivalent to the previous ones.*

Finally, we have:

Definition ([RV, chapter 4]). *In a weakly cartesian closed ∞-cosmos, $f : A \to B$ has a limit if and only if $\Delta \downarrow f$ has a terminal element $(\ell, \lambda) : 1 \to \Delta \downarrow f$, this data defining the limit cone.*

Proposition ([RV, chapter 4]). *In a weakly cartesian closed ∞-cosmos, this definition is equivalent to the previous one.*

Step 3: Choose your favorite definition and prove that the limit exists, perhaps by passing to a biequivalent ∞-cosmos and then replacing your functor with an equivalent one. In this step it's likely that "analytic" techniques might help.

For instance, if your ∞-cosmos is biequivalent to $q\mathcal{C}at$, you can prove that $\Delta \downarrow f$ has a terminal element by finding a vertex t that has the following lifting property for all $n \geq 1$:

$$\begin{tikzcd} \Delta[0] \arrow[r, "i_n"'] \arrow[rr, bend left, "t"] & \partial\Delta[n] \arrow[r] \arrow[d, hook] & \Delta \downarrow f \\ & \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

Step 4 (optional): if you'd prefer to work with the limit of $f : A \to B$ in a different ∞-cosmos then the one where you proved it exists, you can use a biequivalence of ∞-cosmoi to transfer it back. To illustrate, suppose that $f : A \to B$ lives in $\mathcal{K}$ and you proved that it had a limit by passing through a biequivalence $F : \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ and then replacing $Ff : FA \to FB$ with an equivalent functor $f' : A' \to B'$ before constructing the limit:

$$\begin{tikzcd} & 1 \arrow[dr, dashed, "\ell'"] \arrow[d, phantom, "\Downarrow\lambda'"] & \\ A' \arrow[ur, "!"] \arrow[rr, "f'"'] & {} & B' \end{tikzcd}$$

One of the properties of a biequivalence is the following:

Proposition ([RV, chapter 13]). *Consider any cosmological biequivalence $F : \mathcal{K} \to \mathcal{L}$.*

(i) The biequivalence F preserves, reflects, and creates equivalences between ∞-categories, and induces a bijection between equivalence classes of objects.

(ii) *The biequivalence F induces local bijections between isomorphism classes of functors extending the bijection of (i): choosing any pairs of objects $A, B \in \mathscr{K}$ and $A', B' \in \mathscr{L}$ and equivalences $FA \simeq A'$ and $FB \simeq B'$, the map*

$$\mathsf{hFun}(A, B) \xrightarrow{\sim} \mathsf{hFun}(FA, FB) \xrightarrow{\sim} \mathsf{hFun}(A', B') \tag{1}$$

defines a bijection between isomorphism classes of functors $A \to B$ in $\mathscr{K}$ and isomorphism classes of functors $A' \to B'$ in $\mathscr{L}$.

(iii) *The biequivalence F induces local bijections between 2-cells with specified boundary extending the bijections of (i) and (ii): choosing any pairs of objects $A, B \in \mathscr{K}$ and $A', B' \in \mathscr{L}$, equivalences $FA \simeq A'$ and $FB \simeq B'$, functors $f, g : A \rightrightarrows B$ and $f', g' : A' \rightrightarrows B'$, and natural isomorphisms*

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\cong\alpha"] & FB \arrow[d, "\wr"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "Fg"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\cong\beta"] & FB \arrow[d, "\wr"] \\ A' \arrow[r, "g'"'] & B' \end{tikzcd}$$

the map (1) *induces a bijection between 2-cells $f \Rightarrow g$ in $\mathscr{K}$ and 2-cells $f' \Rightarrow g'$ in $\mathscr{L}$.*

Using the property (ii) and the equivalence $FA \simeq A'$, the limit $\ell' : 1 \to A'$ in $\mathscr{L}$ can be lifted to an element $\ell : 1 \to A$ in $\mathscr{K}$. Similarly, using the property (ii), the 2-cell $\lambda' : \ell'! \Rightarrow f'$ can be lifted to $\lambda : \ell! \Rightarrow f$. Now liberal use of all three parts of this proposition can be used to directly prove that the lifted diagram

$$\begin{tikzcd} & 1 \arrow[dr, dashed, "\ell"] \arrow[d, phantom, "\Downarrow\lambda"] & \\ A \arrow[ur, "!"] \arrow[rr, "f"'] & {} & B \end{tikzcd}$$

is a pointwise right extension. Alternatively, a more powerful theorem about the model-independence of ∞-category theory would just tell you that the property of being a pointwise right Kan extension is preserved, reflected, and created.

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