GALOIS EXTENSIONS IN CHROMATIC HOMOTOPY THEORY

Abstract. Notes for the talk by Lior Yanovski at the Viva Talbot! workshop in June 2021.

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1. Higher algebra

Like classical algebra is built from considering the symmetric monoidal abelian category $(\mathrm{Ab}, \otimes, \mathbb{Z})$ of abelian groups, the world of higher algebra can be said to be built out of the stable ∞-category $(\mathrm{Sp}, \otimes, \mathbb{S})$. Just as rings can be defined as algebras over $\mathbb{Z}$, we can define $\mathbb{E}_1$- and $\mathbb{E}_\infty$-rings (what would correspond to associative and commutative rings, respectively) as algebras over the sphere spectrum $\mathbb{S}$. The world of classical algebra can be embedded into higher algebra via the Eilenberg–Mac Lane functor

$$H : \mathrm{Ab} \longrightarrow \mathrm{Sp}$$

which takes an abelian group A the spectrum HA whose homotopy groups is A concentrated in degree 0. This functor is not strictly symmetric monoidal, but it is lax symmetric monoidal. Hence associative and commutative rings are sent to $\mathbb{E}_1$- and $\mathbb{E}_\infty$-rings, respectively. In between classical algebra and higher algebra we can also find the world of homological algebra in the sense that we have an equivalence of stable ∞-categories

$$\mathcal{D}\mathbb{Z} \simeq \mathrm{Mod}_{H\mathbb{Z}}(\mathrm{Sp})\,.$$

There are lots of examples of (ring) spectra such as: the sphere spectrum $\mathbb{S}$, Eilenberg–Mac Lane spectra HR, the spherical group ring of a topological group $\mathbb{S}[G]$, various flavours of K-theory such as KU and KO, and so on.

2. Chromatic homotopy theory

When solving mathematical problems, a common strategy is to divide the problem into smaller pieces. That is, we localise, study the localised picture, and then try to puzzle the bits together into a global picture. Let us work with some p-local spectrum X. The localisation

$$X[p^{-1}] = \mathrm{colim}(X \xrightarrow{p} X \xrightarrow{p} X \xrightarrow{p} \cdots)$$

captures thing that happen away from p, while the completion

$$X_p^\wedge = \lim(\cdots \to X/p^3 \to X/p^2 \to X/p)$$

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captures what happens at p. We can then piece together the information using the arithmetic pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X[1/p] \arrow[d] \\ X_p^\wedge \arrow[r] & X_p^\wedge[1/p] \end{tikzcd}$$

One crucial difference between classical algebra and higher algebra is once we complete at p, there is still non-trivial localisations. That is, for every p-complete spectrum X, there is an asymptotically defined endomorphism

$$v_1 : \Sigma^d X \to X\,.$$

These maps suffices for the definition of localization and completion:

$$X[v_1^{-1}] \in \mathrm{Sp}_{T(1)} \quad \text{and} \quad X_{v_1}^\wedge \in \mathrm{Sp}_{(p,v_1)}^\wedge\,,$$

analogously to before. Again, we have a pullback square, now referred to as the chromatic pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X[v_1^{-1}] \arrow[d] \\ X_{v_1}^\wedge \arrow[r] & X_{v_1}^\wedge[v_1^{-1}] \end{tikzcd}$$

This process continues with endomorphisms v_2, v_3, and so on. In particular, we will write

$$L_{T(n)}X = X_{(p,v_1,\ldots,v_{n-1})}^\wedge[v_n^{-1}] \in \mathrm{Sp}_{T(n)}\ .$$

One way to think about this picture is that we have access to more prime fields. While the finite fields $\mathbb{F}_p$ and the rationals $\mathbb{Q}$ are the only prime fields in algebra, corresponding to the characteristics p and 0, there are more prime fields in higher algebra. For each p, working with spectra there is instead a whole hierarchy of "prime field" interpolating between $\mathbb{Q}$ and $\mathbb{F}_p$.

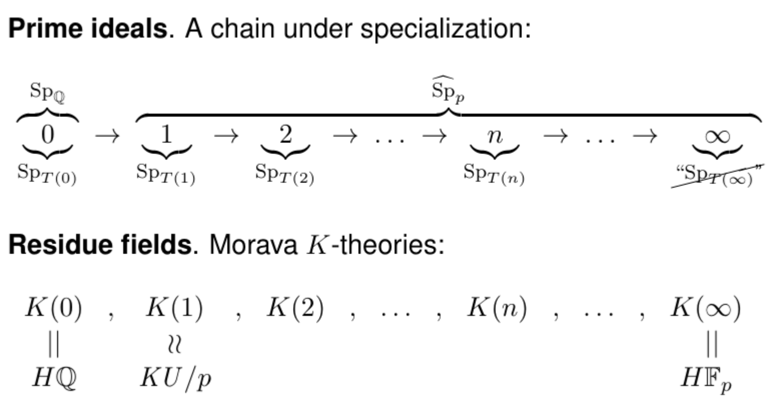

Figure 1. The chromatic picture with the prime ideals and the residue fields in higher algebra.

It is worth noting that the $K(n)$-local world in always contained in the $T(n)$-local world, in the sense that

$$\mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}_{T(n)}$$

for all $0 \leq n < \infty$. The famous telescope conjecture deals with the supposed equivalence between these two world.

Conjecture 1 (Telescope conjecture)**.**

$$L_{T(n)} \simeq L_{K(n)}$$

The telescope conjecture is known to be true for $n = 0$ and $n = 1$. While the telescope conjecture is generally believed to be false, it is here important to note that both the $K(n)$-localised and the $T(n)$-localised world are interesting in their own right, for their own reasons.

- The $K(n)$-local picture in generally easier to work with and more amenable to computation, with important connections to formal groups and algebraic geometry.
- The $T(n)$-local picture is less computable, but carries important information related to unstable homotopy theory (by work of Heuts and Bousfield –Kuhn), as well as redshift phenomena in algebraic K-theory.

3. Galois extensions of commutative rings

Definition 2 (Auslander–Goldman, Rognes)**.** Let $A \to B$ be a homomorphism of commutative ring, making B into an A-algebra, and let G be a finite group acting on B via A-algebra homomorphisms. We say that $A \to B$ is a G**-Galois extension** if the two maps

$$i : A \longrightarrow B^G$$

and

$$h : B \otimes_A B \longrightarrow \prod_{g \in G} B\,, \quad x \otimes y \mapsto (xg(y))_{g \in G}$$

are isomorphisms[1].

The underlying motivation behind the definition is that Galois extensions should correspond to Spec(B) being a principal G-bundle over Spec(A). The reader can check for themselves that this agrees with the classical definition of a Galois extension whenever A and B are fields.

Theorem 3 (Rognes)**.** *If A is a connective commutative ring spectrum, Galois extensions of A are in bijection with Galois extensions of $\pi_0(A)$.*

4. Classical Cyclotomic extensions

Recall that the cyclotomic polynomials are defined recursively via the formula

$$(t^n - 1) = \prod_{d|n} \Phi_d(t)\,.$$

A root ω_m of $\Phi_m(t)$ is necessarily a primitive mth root of unity, and we say that

$$R[\omega_m] = R[t]/\Phi_m(t)$$

is a **cyclotomic extension** of R. Note that $(\mathbb{Z}/m)^\times$ acts on $R[\omega_m]$ by permuting the mth roots of unity. While $R[\omega_m]$ is not in general a $(\mathbb{Z}/m)^\times$-Galois extension of R, it is so whenever m is invertible in R.

Example 4. It is well known that $\mathbb{R}[\omega_4] = \mathbb{C}$ is a C_2-Galois extension of $\mathbb{R}$.

Example 5. We have that $\mathbb{F}_p[\omega_{p^d-1}]$ is a $(\mathbb{Z}/(p^d-1))^\times$-extension of $\mathbb{F}_p$. This splits into a product

$$\mathbb{F}_p[\omega_{p^d-1}] \cong \mathbb{F}_{p^d} \times \cdots \times \mathbb{F}_{p^d}$$

of copies of the $\mathbb{Z}/d$-Galois extension $\mathbb{F}_p \to \mathbb{F}_{p^d}$. The above example can also be lifted integrally into a Galois extension

$$\mathbb{Z}_p[\omega_{p^d-1}] \cong W(\mathbb{F}_{p^d}) \times \cdots \times W(\mathbb{F}_{p^d})$$

splitting into a product of copies of the $\mathbb{Z}/d$-Galois extension $\mathbb{Z}_p \to W(\mathbb{F}_{p^d})$. After inverting p, we have that

$$W(\mathbb{F}_{p^d})[p^{-1}] = \mathbb{Q}_p(\omega_{p^d-1})$$

[1] In the case of classical commutative rings this definition is due to Auslander–Goldman, while Rognes extended it also to commutative ring spectra. In this case "isomorphism" should be interpreted as "equivalence" and the G-fixed points appearing in the map i is to be interpreted as the G-homotopy fixed points B^{hG}. The two maps

$$i : A \to B^{hG} \quad \text{and} \quad h : B \otimes_A B \to \prod_G B \simeq F(G_+, B)$$

cannot then technically be described in terms of elements, but must be constructed as certain right adjoints. Rognes' definition also makes sense more generally for G that might be be finite, namely stably dualizable groups.

which assemble into a maximal unramified extension

$$\mathbb{Q}_p^{\mathrm{ur}} = \bigcup_d \mathbb{Q}_p(\omega_{p^d-1})$$

of $\mathbb{Q}_p$. Moreover, as p is invertible, we also have cyclotomic extensions of p-power order which assemble into

$$\mathbb{Q}_p(\omega_{p^\infty}) = \bigcup_r \mathbb{Q}_p[\omega_{p^r}]\,.$$

Theorem 6 (Kronecker–Weber)**.** *All abelian Galois extensions of $\mathbb{Q}_p$ can be obtained in this way, in the sense that we have an isomorphism*

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{\mathrm{ab}} \cong \hat{\mathbb{Z}} \times \mathbb{Z}_p^\times\,.$$

Here $\hat{\mathbb{Z}}$ corresponds to $\mathbb{Q}_p^{\mathrm{ur}}$ and $\mathbb{Z}_p^\times$ corresponds to $\mathbb{Q}_p(\omega_{p^\infty})$. Note that the Prüfer ring $\hat{\mathbb{Z}}$ is also secretly the Galois group of the Galois extension $\mathbb{F}_p \to \bar{\mathbb{F}}_p$

5. ∞-categorical cyclotomic extensions

To be able to generalise to the setting of higher algebra it will be useful to have an alternative view on cyclotomic extensions which does not make use of quotients. To start with, let us note that for every m we have an isomorphism

$$R[t]/(t^m - 1) \cong R[C_m]$$

where the right hand side is acted on by $(\mathbb{Z}/m)^\times$. For $m = p^r$ consider the short exact sequence

$$0 \longrightarrow C_p \longrightarrow C_{p^r} \longrightarrow C_{p^{r-1}} \longrightarrow 0$$

which induces maps

$$C_p \xrightarrow{\iota} R[C_{p^n}] \to R[C_{p^{r-1}}]\,.$$

If p is invertible in R, then the element

$$\epsilon = \frac{1}{p}\sum_{g\in C_p} \iota(g)$$

in $R[C_{p^n}]$ is an idempotent, so induces a splitting

$$R[C_{p^r}] \cong R[C_{p^{r-1}}] \times R[\omega_{p^r}]\,,$$

with a cyclotomic extension appearing in the second factor on the right hand side.

By an observation by Waldhausen-Schwänzl we can do everything ∞-categorically.

Example 7. We can define spherical Witt vectors in such a way that we have a $\mathbb{Z}/d$-Galois extension

$$\mathbb{S}_p^\wedge \longrightarrow \mathbb{S}W(\mathbb{F}_{p^d})$$

lifting the ordinary $\mathbb{Z}/d$-Galois extension $\mathbb{Z}_p \to W(\mathbb{F}_{p^d})$.

Remark 8. An generalization of the above ideas, which allows for adjoining roots of arbitrary elements $a \in \pi_0 R^\times$:

$$\pi_0(R[a^{1/n}]) \simeq \pi_0(R)[t]/(t^n - s)$$

via twisted group rings, was also considered by Lawson.

6. Chromatic Galois extensions

Rognes' framework for Galois extensions also makes sense in localised settings as well as for groups other than finite one. In particular, in $K(n)$-local spectra we have a (pro)-Galois extension

$$L_{K(n)}\mathbb{S}_p^\wedge \longrightarrow E_n$$

where E_n denotes Morava E-theory. Here the (pro)-Galois group is the so called Morava stablizer group $\mathbb{G}_n = \hat{\mathbb{Z}} \ltimes S_n$, where S_n denotes the automorphism group of the height n Honda formal group law over $\mathbb{F}_p$. That this is indeed a Galois extension is essentially a theorem by Devinatz–Hopkins, reworked by Rognes into the Galois extension formalism. Moreover, this is the maximal Galois extensions in some precise sense.

One can think of S_n as a twisted form of $GL_n(\mathbb{Z})^n$. The abelianisation of the Morava stablizer group is again the group appearing in the Kronecker–Weber theorem:

$$\mathbb{G}_n^{\mathrm{ab}} \simeq \hat{\mathbb{Z}} \times \mathbb{Z}_p^{\times} .$$

The projection onto the second factor is given by the so-called determinant map

$$\det : S_n \to \mathbb{Z}_p^{\times} .$$

One can think of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q})$ as the height 0 analogue of the Morava stabilizer group. We might wonder whether you can realize those abelian Galois extensions again as some sort of cyclotomic extensions.

- For the $\hat{\mathbb{Z}}$ part, the answer is yes. The finite quotients

  $$\mathbb{G}_n \to \hat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/d$$

  classify the Galois extensions

  $$L_{K(n)}\mathbb{S}_p^{\wedge} \longrightarrow L_{K(n)}\mathbb{S}W(\mathbb{F}_{p^d})$$

  and hence embed into prime to p cyclotomic extensions.
- However, the finite quotients

  $$\mathbb{G}_n \to \mathbb{Z}_p^{\times} \twoheadrightarrow (\mathbb{Z}/p^r)^{\times}$$

  do not classify p-power cyclotomic extension.

There is actually a stronger no-go theorem.

Theorem 9 (Devalapurkar)**.** *For $n \geq 1$, there is no $K(n)$-local commutative ring spectrum R such that $\pi_0 R$ contains a primitive pth root of unity.*

However, they do come from some some sort of higher analogue of cyclotomic extensions!

7. Higher Roots of Unity

From the functor of points point of view: for commutative R-algebras S, the R-algebra $R[C_{p^n}]$ corepresents p^rth roots of unity is the sense that we have a bijection

$$\mathrm{Hom}_R(R[C_{p^r}], S) \cong \mathrm{Hom}(C_{p^r}, S^{\times}) .$$

In the same way, $R[\omega_{p^r}]$ corepresents primitive p^rth roots of unity.

In the higher setting, we define the height n root of unity of S as a map

$$\omega^{(n)} : HC_{p^r} \longrightarrow \Omega^n S^{\times} .$$

This is corepresented by $R[B^n C_{p^n}]$ and acted on by $(\mathbb{Z}/p^r)^{\times}$. There is also a natural definition for $\omega^{(n)}$ to be primitive. It turns our that working $K(n)$-locally, the ring $\mathbb{S}_{K(n)}[B^n C_{p^n}]$ splits equivariantly as

$$\mathbb{S}_{K(n)}[B^n C_{p^n}] \simeq \mathbb{S}_{K(n)}[B^n C_{p^{r-1}}] \times \mathbb{S}_{K(n)}[\omega_{p^r}^{(n)}] .$$

Here, the second factor corepresents the primitive higher roots of unity.

Theorem 10 (Westerland)**.** *The algebra $\mathbb{S}_{K(n)}[\omega_{p^r}^{(n)}]$ is $(\mathbb{S}/p^r)^{\times}$-Galois and is classified by the composition*

$$\mathbb{G}_n \xrightarrow{\det} \mathbb{Z}_p^{\times} \twoheadrightarrow (\mathbb{Z}/p^r)^{\times} .$$

The slogan here is that the determinant $\det : \mathbb{G}_n \longrightarrow \mathbb{Z}_p^{\times}$ is the p-adic cyclotomic character.

8. Higher semiadditivity (aka ambidexterity)

Given $m \geq -2$, we say that a space A is called m-finite if it is m-truncated, has finitely many connected components and all of its homotopy groups are finite. It is called π-finite if it is m-finite for some m.

Definition 11 (Hopkins–Lurie)**.** For every space A, there is a canonical map

$$\operatorname*{colim}_A \longrightarrow \lim_A .$$

We say that an ∞-category is ∞**-semiadditive** if this map is an equivalence for all π-finite spaces A.

Semiadditivity gives us access to a bunch of important structure. In particular, semiadditivity gives us the ability to sum a finite family of morphisms between two objects. That is, it tells us how to define integration on a morphism $\varphi : A \to \mathrm{Map}(X, Y)$ where A is π-finite. This is map from X to Y denoted

$$\int_A \varphi .$$

In particular, given a π-space we always have a distinguished map $|A| : \mathbb{S} \to X$ obtained by integrating the constant map $A \to \mathrm{Map}(X, X)$ at the identity Id_X.

Theorem 12 (Hopkins–Lurie, Greenlees–Hovey–Sadofsky)**.** *The ∞-categories $\mathrm{Sp}_{K(n)}$ are ∞-semiadditive for all $0 \leq n < \infty$.*

It turns out that in $\mathrm{Sp}_{K(n)}$, the map $|B^n C_p| : X \to X$ is always invertible in $\pi_0(R)$. We consider the fibre sequence

$$B^n C_p \longrightarrow B^n C_{p^n} \longrightarrow B^n C_{p^{r-1}}$$

which induces a map

$$\iota : B^n C_p \longrightarrow \Omega^\infty(\mathbb{S}_{K(n)}[B^n C_{p^r}]) .$$

The integration-framework allows us to define an idempotent

$$\epsilon = \frac{1}{|B^n C_p|} \int_{g \in B^n C_p} \iota(g)$$

in $\pi_0(\mathbb{S}_{K(n)}[B^n C_{p^r}])$. Being an idempotent, it induces a $(\mathbb{Z}/p^r)^\times$-equivariant splitting of the above fibre sequence in such a way that

$$\mathbb{S}_{K(n)}[B^n C_{p^r}][(1-\epsilon)^{-1}] \simeq \mathbb{S}_{K(n)}[\omega_{p^r}^{(n)}] .$$

So, $\mathbb{S}_{K(n)}[\omega_{p^r}^{(n)}]$ corepresents the space of primitve p^rth roots of unity of height n.

9. Telescopic Galois Extensions

Theorem 13 (Carmeli–Schlank–Yanovski)**.** *The ∞-category $\mathrm{Sp}_{T(n)}$ is ∞-semiadditive of height n for all $0 \leq n < \infty$.*

Theorem 14. *The algebra $\mathbb{S}_{T(n)}[\omega_{p^r}^{(n)}]$ is a $(\mathbb{Z}/p^r)^\times$-Galois extension of $\mathbb{S}_{T(n)}$.*

Corollary 15. *We can lift every finite abelian G-Galois extension $\mathbb{S}_{K(n)} \to R$ to a G-Galois extension $\mathbb{S}_{T(n)} \to R^f$.*