Real bordisms, Real orientations, and Lubin–Tate spectra

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Abstract

In this talk, I will discuss the Real bordism spectrum and the theory of Real orientations. This is an equivariant refinement of the complex cobordism spectrum and the theory of complex orientations. The Real bordism spectrum and its norms are crucial in Hill–Hopkins–Ravenel's solution of the Kervaire invariant one problem in 2009. I will talk about their solution and explain how the Real bordism spectrum is further creating many connections between equivariant stable homotopy theory and chromatic homotopy theory. These newly established connections allow one to use equivariant machinery to attack classical computations that were long considered unapproachable. This talk contains joint work with Agnès Beaudry, Jeremy Hahn, Mike Hill, Guchuan Li, Lennart Meier, Guozhen Wang, Zhouli Xu, and Mingcong Zeng.

Contents

*Typed by Jack Davies (Universiteit Utrecht) and Heyi Zhu (University of Ullinois Urbana-Champaign).

1 What does Real mean?

Let C_2 be the cyclic group of order 2 and write τ for the generator. A *Real space*[1] is simply a space X with a C_2-action, so X comes with a map $\tau\colon X \to X$ such that $\tau \circ \tau$ is the identity. A *Real vector bundle* E over a Real space X is the data of a complex vector bundle $E \to X$ and a Real space structure on E, such that $E \to X$ is C_2-equivariant and the C_2-action on all fibres

$$\tau\colon E_x \to E_{\tau(x)}$$

is *anti* $\mathbf{C}$*-linear*, meaning that $\tau(z \cdot v) = \overline{z} \cdot \tau(v)$, where $\overline{z}$ is the complex conjugate of $z \in \mathbf{C}$.[2] As in the nonequivariant world, we can form the monoid of isomorphism classes of Real vector bundles over X and take its Grothendieck group completion, and we obtain a group $K_{\mathbf{R}}(X)$. This is a cohomology theory, and leads to the C_2-equivariant spectrum $K_{\mathbf{R}}$ called *Atiyah's Real K-theory*; see [Ati66].

To property study equivariant cohomology theories like $K_{\mathbf{R}}$, we use *equivariant stable homotopy theory.* Recall that for a nonequivariant spectrum X, its homotopy groups can be given by $\pi_n X = [S^n, X]$. One main feature of equivariant stable homotopy theory (for some fixed group G) is that there are more spheres. For every G-representation V, we obtain a sphere S^V by one-point-compactification. This also yields more homotopy groups for G-spectra using the formula $\pi_V^G X = [S^V, X]^G$, which leads to the $RO(G)$-graded homotopy groups of X, denoted by $\pi_\star^G X$; see [LMS86] or [HHR16]. Let us talk about two examples of such theories.

Example 1 (Atiyah's Real K-theory $K_{\mathbf{R}}$)*.* The C_2-spectrum $K_{\mathbf{R}}$ combines complex K-theory KU and real K-theory KO. The underlying nonequivariant spectrum is KU, and the (homotopy) fixed point spectrum is KO:

$$K_{\mathbf{R}}^{C_2} \simeq K_{\mathbf{R}}^{hC_2} \simeq KO$$

There are also two different periodicities in the $RO(C_2)$-graded homotopy groups of $K_{\mathbf{R}}$. The first is an equivariant refinement of complex Bott periodicity

$$\pi_{\star+\rho}^{C_2} K_{\mathbf{R}} = \pi_\star^{C_2} K_{\mathbf{R}},$$

where ρ indicates the regular representation of C_2 (which can also be given by $\mathbf{C}$ with the complex conjugation action), which has dimension 2. There is also an 8-fold period from real Bott periodicity

$$\pi_{\star+8}^{C_2} K_{\mathbf{R}} = \pi_\star^{C_2} K_{\mathbf{R}}.$$

Before we get into the more important example for us, let us recall its nonequivariant counterpart.

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[1] As mentioned in [Ati66], the motiviation for the adjective *Real* with a capital "r" comes from real algebraic geometry: if X is the set of complex points of an algebraic variety over $\mathbf{R}$, then X has the natural structure of a Real space in the sense discussed here, with C_2-action given by complex conjugation.

[2] It's important to note that this is not equivalent to the data of a C_2-equivariant complex vector bundle! In that case we would ask τ to be $\mathbf{C}$-linear on fibres, so $\tau(z \cdot v) = z \cdot \tau(v)$.

Example 2 (Complex bordism MU). Write γ_n for the universal complex n-bundle over $BU(n)$, and $BU(n)^{\gamma_n}$ for its Thom space. There is a map $BU(n) \to BU(n+1)$ classified by sending γ_n to the complex $(n+1)$-bundle given by adding a trivial line bundle. This yields a map between Thom spaces

$$\Sigma^2 BU(n)^{\gamma} \simeq BU(n)^{\gamma_n \oplus \mathbf{C}} \to BU(n+1)^{\gamma_{n+1}},$$

and these maps give us structure maps for the nonequivariant spectrum MU of *complex bordism.*

Everything we just did for MU has a natural C_2-equivariant refinement.

Example 3 (Real bordism $MU_{\mathbf{R}}$). Both $BU(n)$ and γ_n are naturally Real spaces by complex conjugation, and the map $\gamma_n \to BU(n)$ defines a Real vector bundle. We can also check that the map $BU(n) \to BU(n+1)$ is C_2-equivariant, where we now view the trivial line bundle over $BU(n)$ has as a copy of the regular representation ρ. This yields the C_2-equivariant map of spaces

$$\Sigma^{\rho} BU(n)^{\gamma} \simeq BU(n)^{\gamma_n \oplus \mathbf{C}} \to BU(n+1)^{\gamma_{n+1}},$$

giving us structure maps for the C_2-equivariant spectrum $MU_{\mathbf{R}}$ of *Real bordism.* This has underlying spectrum MU by construction, but its (homotopy) fixed points are something much more complicated than MO (real bordism), although they are known. This spectrum $MU_{\mathbf{R}}$ is crucial in Hill–Hopkins–Ravenel's solution to the Kervaire invariant one problem (more on this later!).

The classical spectrum MU has many desirable features. Its homology groups $MU_*(X)$ are the groups of bordism classes of manifolds over X with complex linear structure on their stable normal bundle, ie, bordism classes of almost complex manifolds over X. Moreover, its homotopy groups are an infinite polynomial ring

$$\pi_* MU = MU_*(\mathrm{pt}) \simeq \mathbf{Z}[x_1, x_2, \ldots] \qquad |x_i| = 2i,$$

from calculations done by Milnor and Novikov; also see [Ada74, §II]. There is also the theory of complex orientations; see *ibid.*

Definition 4 (Complex orientations)**.** Given a multiplicative cohomology theory E, then a *complex orientation* on E is a class $x \in \widetilde{E}^2(\mathbf{CP}^\infty)$ which restricts to a unit inside

$$\widetilde{E}^2(\mathbf{CP}^1) = \widetilde{E}^2(S^2) = \widetilde{E}^0(S^0) = \pi_0 E.$$

Many calculations simplify if E has a complex orientation. For example, using the Atiyah–Hirzebruch spectral sequence one can calculate

$$E^*(\mathbf{CP}^\infty) \simeq E^*[\![x]\!] \qquad E^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \simeq E^*[\![x,y]\!],$$

where x, y are classes in degree 2. There is a map $\mathbf{CP}^\infty \times \mathbf{CP}^\infty \to \mathbf{CP}^\infty$ which classifies the tensor product of line bundles, and this induces a map on E-cohomology

$$E^*(\mathbf{CP}^\infty) \to E^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty).$$

If E is complex oriented, then we can consider the image of $x \mapsto F(x, y)$, some power series in two variables. This power series actually defines what is called a *formal group law* over $\pi_* E$. One can show that for a multiplicative cohomology theory E, the set of complex orientations of E are in bijection with the homotopy classes of maps $MU \to E$ of homotopy commutative ring spectra; see [Lur, Lec.6]. This can be reformulated to the following theorem of Quillen.

Theorem 5 (Quillen)**.** *The universal complex oriented spectrum is MU, and it carries the universal formal group law.*

There are other (simpler) examples of complex oriented ring spectra.

Example 6 (Eilenberg–Mac Lane spectra). Writing $H\mathbf{Z}$ for the Eilenberg–Mac Lane spectrum for the integers, the spectrum representing integral singular cohomology, then there is a map $MU \to H\mathbf{Z}$. This can either be seen using Definition 4 or by setting $MU \to H\mathbf{Z}$ to be the zeroth truncation. The formal group law associated to $H\mathbf{Z}$ is called the *additive formal group law*, and is given by

$$\mathbf{G}_a(x, y) = x + y.$$

This reflects the usual additivity properties of Chern classes in singular cohomology.

Example 7 (Complex K-theory). Another typical example is KU. Constructing the complex orientation can be obtained through a study of KU, and the associated formal group law is called the *multiplicative formal group law*, given by the formula

$$\mathbf{G}_m(x, y) = x + y - \beta xy,$$

where $\beta \in \pi_2 KU$ is the Bott class.

There is a natural C_2-equivariant lift of the story of complex orientations to what are called *Real orientations*; see [Ara79] and [HK01].

Definition 8 (Real orientations)**.** Given a multiplicative C_2-equivariant cohomology theory E, then a *Real orientation* on E is a class $x \in \widetilde{E}^{\rho}_{C_2}(\mathbf{CP}^\infty)$ which restricts to a unit inside

$$\widetilde{E}^{\rho}_{C_2}(\mathbf{CP}^1) = \widetilde{E}^{\rho}_{C_2}(S^\rho) = \widetilde{E}^{0}_{C_2}(S^0) = \pi_0^{C_2} E.$$

Here, and everywhere else in the lecture, $\mathbf{CP}^n$ and $\mathbf{CP}^\infty$ will always have the structure of a Real space given by complex conjugation.

Araki tells us that given a Real oriented C_2-spectrum E, then we can calculate some C_2-equivariant $RO(C_2)$-graded cohomology groups

$$E^\star_{C_2}(\mathbf{CP}^\infty) \simeq E^\star_{C_2}[\![\overline{x}]\!] \qquad E^\star_{C_2}(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \simeq E^\star_{C_2}[\![\overline{x}, \overline{y}]\!],$$

where $\overline{x}$ and $\overline{y}$ are now classes in degree ρ.[3] As you should expect by now, the map $\mathbf{CP}^\infty \times \mathbf{CP}^\infty \to \mathbf{CP}^\infty$ given above is C_2-equivariant, and we obtain a formal group law, now over $\pi^{C_2}_\star E$. Moreover, Hu and Kriz show that homotopy classes of maps of homotopy commutative C_2-ring spectra $MU_{\mathbf{R}} \to E$ are in bijection with Real orientations of E. Prime examples of Real oriented theories are of course Atiyah's Real K-theory $K_{\mathbf{R}}$, and Real bordism itself $MU_{\mathbf{R}}$. But there are also more examples, which we will explore now.

[3] The notation $\overline{x}$ is supposed to remind us that this element is a C_2-equivariant lift or refinement of the classical class x which has degree 2, the underlying dimension of ρ.

2 Lubin–Tate spectra

It turns out that KU belongs for a more general class of spectra; see [Lur, Lec.21-2]. Fix a perfect field κ of characterisic $p > 0$, and a formal group law Γ_n of height n over κ. The deformation theory of Lubin–Tate says Γ_n admits a universal deformation, which can be characterised by a map $MU_* \to E_{n*}$, where

$$E_{n*} \simeq W(\kappa)[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}]$$

is the universal deformation ring. The Landweber exact functor theorem states that this graded ring is actually the homotopy groups of a homotopy commutative ring spectrum E_n, called the *Lubin–Tate spectrum* of the pair (κ, Γ_n). This is a complex oriented cohomology theory whose associated formal group law is the universal deformation of Γ_n.

Example 9 (p-adic complex K-theory). The connection with KU can be explained as follows: if $\kappa = \mathbf{F}_p$ is the field with two elements and Γ_1 is the multiplicative formal group law $\Gamma_1(x, y) = x + y - \beta xy$, then the associated Lubin–Tate spectrum E_1 is the p-completion of K-theory $KU_p^\wedge$.

Example 10 (Elliptic cohomologies). Another huge source of formal group laws comes from elliptic curves. This leads to elliptic cohomology theories, topological modular forms, the Witten genus, and much more. The Lubin–Tate spectra associated to such formal groups from elliptic curves are forms of elliptic cohomology.

An application of Lubin–Tate spectra is to study the structure of the stable homotopy groups of spheres. Recall the following theorem of Freudenthal from 1937.

Theorem 11 (Freudenthal). *The abelian groups $\pi_{n+k}S^n$ stabilise for $n > k + 1$.*

We then define the *kth stable homotopy group of spheres* by the (eventually stable) colimit

$$\pi_k^{\mathrm{st}} S^0 = \operatorname{colim} \pi_{n+k} S^n.$$

In the 1930s, Pontryagin constructed an isomorphism

$$\pi_k^{\mathrm{st}} S^0 \simeq \Omega_k^{\mathrm{fr}},$$

where Ω_k^{fr} is the group of cobordism classes of stably framed k-manifolds. This establishes a deep relationship between homotopy theory and geometry. Over the past 80 years, describing $\pi_*^{\mathrm{st}} S^0$ has been fundamental to algebraic topology. Lubin–Tate spectra can isolate certains "sectors" in our computations of these homotopy groups, and also give connections to other areas of mathematics (such as modular forms and geometric topology).

To take us back to equivariant stable homotopy theory, we would like to put a group action on these Lubin–Tate spectra E_n. The work of Lubin–Tate shows that the *Morava stabiliser group*

$$\mathbf{G}_n = \mathbf{G}(\kappa, \Gamma_n) = \operatorname{Aut}(\Gamma_n) \rtimes \operatorname{Gal}(\kappa/\mathbf{F}_p)$$

acts on E_{n*}, and one might hope to lift this to an action of spectra on E_n. A solution is provided by the following theorem of Goerss, Hopkins, and Miller using Goerss–Hopkins obstruction theory; see [GH04, §7].

Theorem 12 (Goerss–Hopkins–Miller)**.** *The homotopy commutative ring spectrum E_n has the structure of an $\mathbf{E}_\infty$-ring, which is unique up to contractible choice. Moreover, the $\mathbf{G}(\kappa, \Gamma_n)$-action on E_{n*} lifts to an action on E_n by maps of $\mathbf{E}_\infty$-rings, which is also unique up to contractible choice.*

We can therefore consider E_n as a $\mathbf{G}(\kappa, \Gamma_n)$-equivariant commutative ring spectrum.[4] Let us now fix our prime $p = 2$, and specialise to a simple subgroup of the Morava stabiliser group $C_2 \leqslant \mathbf{G}(\kappa, \Gamma_n)$. The action of C_2 on E_{n*} now comes from the inverse $[-1]_{\Gamma_n}$ of the formal group law Γ_n. The $\mathbf{G}(\kappa, \Gamma_n)$-action on Lubin–Tate spectra restricts to a C_2-action of $\mathbf{E}_\infty$-rings on E_n. Consider the following schematic:

$$\begin{tikzcd} MU_{\mathbf{R}} \arrow[r, "?"] \arrow[d, dotted] & E_n \arrow[d, dotted] \\ MU_* \arrow[r] & E_{n*} \end{tikzcd}$$

We already have the map of rings $MU_* \to E_{n*}$, and moreover we know that it is C_2-equivariant; it classifies a formal group. We also know that both of these rings and their C_2-actions come from the nonequivariant homotopy groups of two C_2-spectra $MU_{\mathbf{R}}$ and E_n. The natural question is then:

Is there a C_2-equivariant map of spectra $MU_{\mathbf{R}} \to E_n$ lifting $MU_* \to E_{n*}$?

A priori, this is an overly optimistic question, as the C_2-action on the left comes from the geometry of complex conjugation, and the C_2-action on the right comes from obstruction theory and the deformation theory from purely algebraic gadgets. However, there is a positive answer, given to us by work of Hahn and the speaker; see [HS20].

Theorem 13 (Hahn–S.)**.** *The Lubin–Tate spectrum E_n is Real oriented: it receives a C_2-equivariant map*

$$MU_{\mathbf{R}} \to E_n$$

from the Real bordism spectrum $MU_{\mathbf{R}}$.

This opens the door to a series of computations in stable homotopy theory.

3 Chromatic homotopy theory

The theory of chromatic homotopy theory tells us that the collection of Lubin–Tate spectra for various heights determines p-local stable homotopy theory; see [Lur] for some general information. The following is the chromatic convergence theory of Hopkins–Ravenel; see [Rav92, §8.6].

Theorem 14 (Hopkins–Ravenel)**.** *The homotopy limit of the diagram*

$$\cdots \to L_{E_n} S^0 \to L_{E_{n-1}} S^0 \to \cdots \to L_{E_1} S^0 \to L_{E_0} S^0$$

is equivalent to the p-local sphere spectrum $S^0_{(p)}$. Here L_E denotes the Bousfield localisation *functor with respect to the spectrum E.*

[4] This is done using a *cofree* construction.

This theorem states that to understand the (p-localisation of the) stable homotopy groups of spheres, one can understand the localisations $L_{E_n}S^0$ and how they fit together. The *chromatic fracture square* states that the following natural diagram of spectra is Cartesian:

$$\begin{tikzcd} L_{E_n}S^0 \arrow[r] \arrow[d] & L_{K(n)}S^0 \arrow[d] \\ L_{E_{n-1}}S^0 \arrow[r] & L_{E_{n-1}}L_{K(n)}S^0 \end{tikzcd}$$

The spectrum $K(n)$ above is that of *Morava K-theory* of height n (at the prime p). The slogan of chromatic homotopy theory then reads:

In order to study $S^0_{(p)}$, we need to study the $K(n)$-local spheres and how they "glue" together.

The connection between chromatic homotopy theory and equivariant homotopy theory lies in the following theorem of Hopkins and Devinaz.

Theorem 15 (Hopkins–Devinaz). *The map of spectra $L_{K(n)}S^0 \to E_n^{h\mathbf{G}_n}$ is an equivalence.*

This implies that the spectra E_n^{hG}, for G a finite[5] subgroup of $\mathbf{G}_n$, are central objects to study in chromatic homotopy theory. In general, such spectra capture large scale periodicity phenomena in stable homotopy theory, and modern detection theorems help us directly study elements in π_*S^0 using the unit map (also known as the Hurewicz map) $\pi_*S^0 \to \pi_*E_n^{hG}$. The latter technique is used in Hill, Hopkins, and Ravenel's solution to the Kervaire invariant one problem (still to come!). Let us discuss some computations at low heights.

Example 16 (At height 1). There is a homotopy fixed point spectra sequence

$$H^*(G; \pi_*E_n) \Longrightarrow \pi_*E_n^{hG}$$

which we totally understand for $n = 1$, as $E_1^{hC_2} \simeq KO_2^\wedge$. This is very closely related to the image of $J\colon \pi_*O \to \pi_*S^0$, which captures everything above the line of slope $1/5$ in the Adams–Novikov spectral sequence for S^0 due to work by Mahowald.

Example 17 (At height 2). The Lubin–Tate spectra at height 2 are closely related to *topological modular forms* tmf (and tmf with level structure); see [Beh20]. These spectra are topological refinements of the classical rings of modular forms in number theory and are closely related to elliptic curves. This is a very active area of research! The Hurewicz map $\pi_*S^0 \to \pi_*\,\mathrm{tmf}$ detects an astounding number of elements, nearly everything in π_*S^0 for $* < 60$. These spectra can also be used to give resolutions of the $K(2)$-local sphere. Computations of height 2 Lubin–Tate spectra rely heavily on the geometry of elliptic curves. One usually choses a nice supersingular[6] elliptic curve which can give an explicit understanding of how G acts on E_{2*}.

[5] The Morava stabiliser group $\mathbf{G}_n$ is a pro-finite group, and therefore studying fixed points with respect to the whole group is difficult.

[6] A *supersingular* elliptic curve is one whose formal group has height 2, otherwise we say an elliptic curve is *ordinary* when the formal group has height 1.

At higher heights the spectra E_n^{hG} see a lot more information, but we find ourselves with many problems. Firstly, the homotopy groups $\pi_* E_n^{hG}$ are extremely difficult to compute. This comes from the fact that the G-action on $\pi_* E_n$ is very difficult to compute, a consequence of this action coming purely from obstruction theory. It is also not possible to use elliptic curves anymore either, as they always have height $\leqslant 2$. Work by Behrens–Lawson on *topological automorphic forms* attempts to find formal groups of higher height using more sophisticated algebro-geometric input; see [BL10].

What is easy to understand is the C_2-action on $MU_{\mathbf{R}}$, as this comes purely from geometric inputs, hence the Real orientations of Lubin–Tate spectra establish the first known connection between these actions. This is how Theorem 13 makes many previously unaccessable computations possible.

(30 minute break)

4 Computations of E_n^{hG}

The Real orientation of Theorem 13 induces a morphism of $RO(C_2)$-graded C_2-homotopy fixed point spectral sequences

$$\mathrm{HFPSS}(MU_{\mathbf{R}}^{hC_2}) \to \mathrm{HFPSS}(E_n^{hC_2}),$$

which converge to the C_2-homotopy fixed points of the above spectra. Using work by Hu and Kriz (see [HK01]), the spectral sequence on the left is completely understood, and the differentials[7]

$$d_{2^{k+1}-1}(u_{2\sigma}^{2^{k-1}}) = \overline{x}_{2^k-1} a_\sigma^{2^{k+1}-1}$$

induce all of the differentials in $\mathrm{HFPSS}(E_n^{hC_2})$. Indeed, we have the following theorem.

Theorem 18 (Hahn–S.)**.** *The E_2-page of the $RO(C_2)$-graded homotopy fixed point spectral sequence of E_n takes the form*

$$E_2^{s,t}(E_n^{hC_2}) \simeq W(\mathbf{F}_{2^n})[\![\overline{u}_1, \ldots, \overline{u}_{n-1}]\!][\overline{u}^{\pm}] \otimes \mathbf{Z}[u_{2\sigma}^{\pm}, a_\sigma]/(2a_\sigma).$$

The classes $\overline{u}_1, \ldots, \overline{u}_{n-1}, \overline{u}^{\pm}$, and a_σ are permantent cycles. All the differentials in the spectral sequence are determined by the differentials

$$d_{2^{k+1}-1}(u_{2\sigma}^{2^{k-1}}) = \overline{u}_k \overline{u}^{2^k-1} a_\sigma^{2^{k+1}-1} \qquad 1 \leqslant k \leqslant n-1,$$

$$d_{2^{n+1}-1}(u_{2\sigma}^{2^{n-1}}) = \overline{u}^{2^n-1} a_\sigma^{2^{n+1}-1} \qquad k = n,$$

and the multiplicative structures.

For $n = 3$, we have the E_3-, E_7-, E_{15}-, and E_∞-pages for the $\mathrm{HFPSS}(E_3^{hC_2})$ can be found as Figure 1, Figure 2, Figure 3, and Figure 4, respectively, all from [HS20].

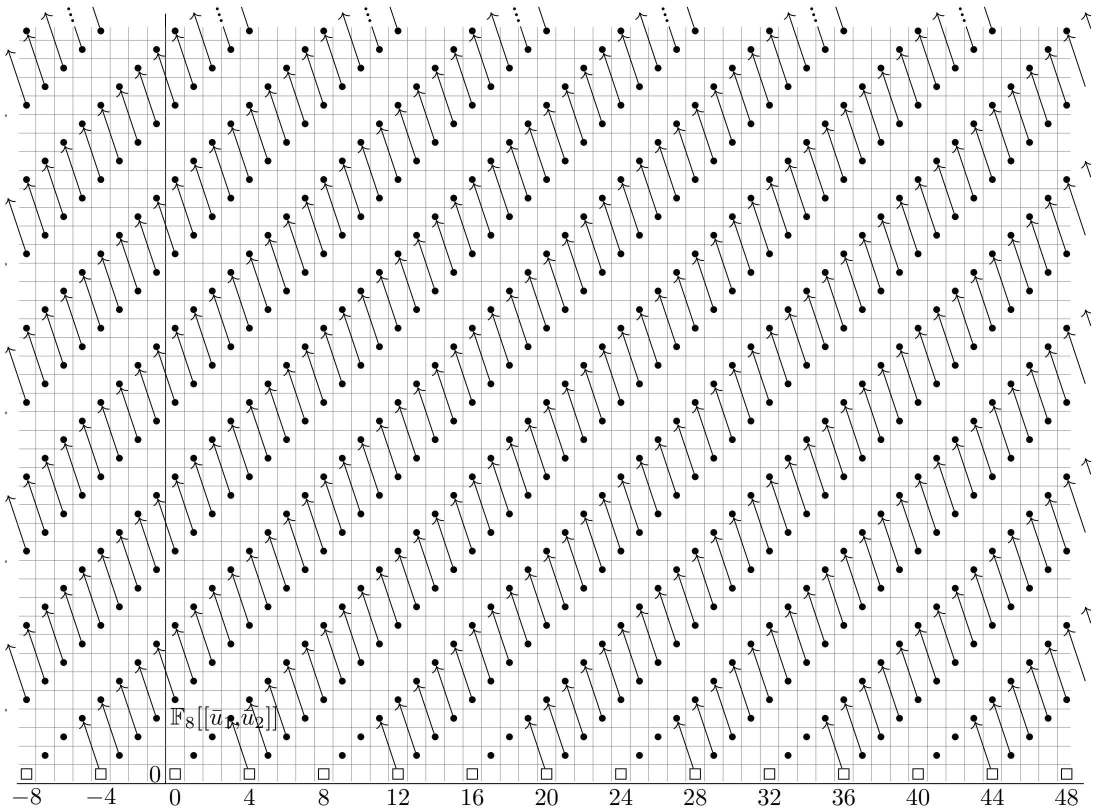

Figure 1: E_3-page of the homotopy fixed point spectral sequence for $E_3^{hC_2}$.

Notice that on the E_∞-page Figure 4, we can visually see that $E_3^{hC_2}$ is 32-periodic.[8] This is actually just one of many general nice properties that these spectra satisfy.

Theorem 19 (Hahn–S.)**.** *The spectra $E_n^{hC_2}$ are 2^{n+2}-periodic for all $n \geqslant 1$. As C_2-spectra, the Lubin–Tate spectra E_n are C_2*-equivariantly even,[9] *meaning that $\underline{\pi}_{k\rho-1}E_n = 0$ for all $k \in \mathbf{Z}$ and $\underline{\pi}_{k\rho}E_n$ is the constant Mackey functor for all $k \in \mathbf{Z}$. The C_2-spectra E_n are also Real Landweber exact, meaning for all C_2-spectra X we have an isomorphism*[10]

$$MU_{\mathbf{R}\bigstar}(X) \otimes_{MU_*} E_{n*} \to E_{n\bigstar}(X).$$

Let us go back to the E_∞-page of $E_3^{hC_2}$ again; see Figure 4. Notice there are classes labelled

[7] The elements $u_{2\sigma}$ and a_σ are described in [HM17, Cor.4.7]. In particular, in the case below, the class a_σ is a permantent cycle, and represents the essential C_2-equivariant map $S^0 \to S^\sigma$ which sends one point to the North pole and the other point to the South pole.

[8] When a ring spectrum E is said to be $2n$*-periodic*, we mean that E is at least $2n$-periodic. For example, we could say KU is 4-periodic, although we never would. However, in all known examples, the periods stated in this talk are sharp.

[9] Recall that nonequivariantly we say a spectrum X is *even* if the odd homotopy groups $\pi_{2k+1}X$ vanish.

[10] See [HM17, §3] for an explanation of C_2-equivariant evenness and Real Landweber exactness.

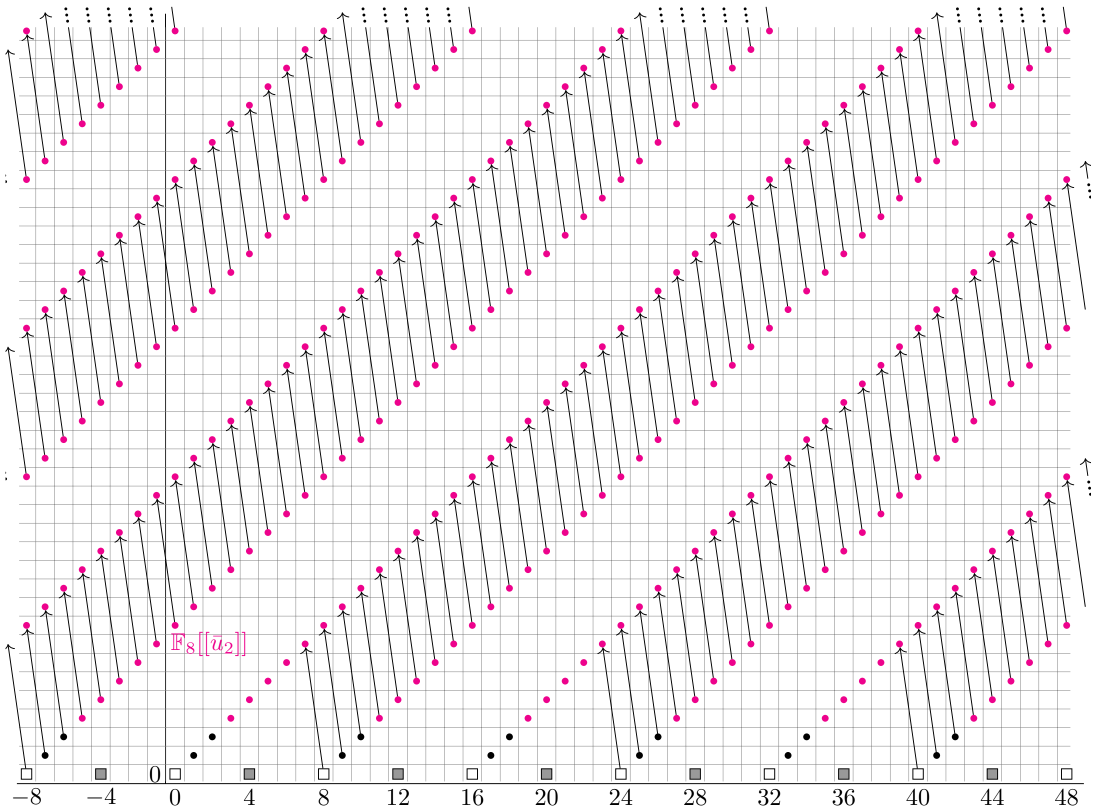

Figure 2: E_7-page of the homotopy fixed point spectral sequence for $E_3^{hC_2}$.

by various Greek letters, such as η, ν, and σ, corresponding to the Hopf elements, as well as more exotic elements such as elements in the $\overline{\kappa}$ family, $\overline{\kappa}$ and $\overline{\kappa}_2$. There is also a general theory developed which tells us how much of the Hurewicz image of $\pi_* S^0$ is detected by various C_2-fixed point spectra; see [LSWX19] and [HS20, §7].

Theorem 20 (Li–S.–Wang–Xu)**.** *The C_2-fixed points of $MU_{\mathbf{R}}$ detects the Hopf-, Kervaire-, and $\overline{\kappa}$-family.*

Theorem 21 (Li–S.–Wang–Xu, Hahn–S.)**.** *The C_2-fixed points of E_n detects the first n elements of the Hopf- and Kervaire-family, and the first $(n-1)$ elements of the $\overline{\kappa}$-family.*

There are also methods to access computations for groups that larger than C_2 using the Hill–Hopkins–Ravenel norm; see [HHR16]. If H is a subgroup of a group G, then there is a functor N_H^G from the category of equivariant H-spectra to that of equivariant G-spectra.[11] If

[11] The norm functor $N_{C_2}^{C_{2^m}}$ is simple to describe: the underlying object $N_{C_2}^{C_{2^m}} X$ is $X^{\otimes 2^{m-1}}$, with the generator of C_{2^m} acting by

$$x_1 \otimes \cdots \otimes x_{2^{m-1}} \mapsto x_2 \otimes \cdots \otimes x_{2^{m-1}} \otimes \tau(x_1).$$

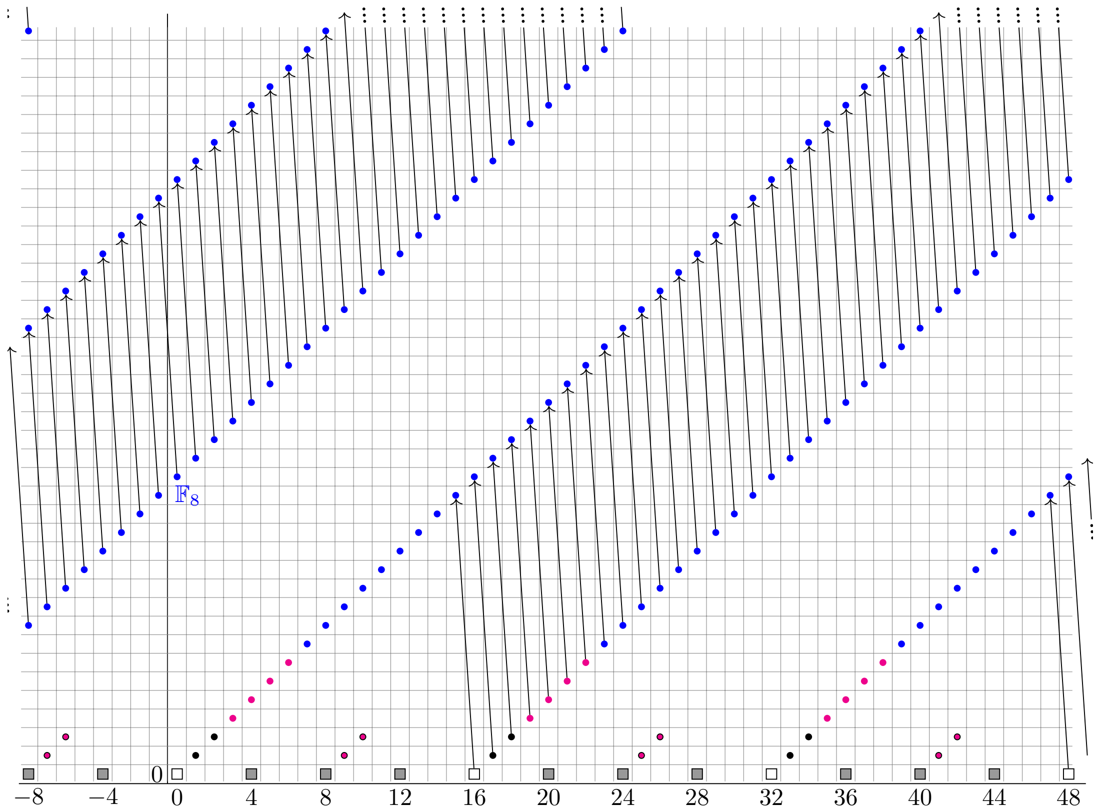

Figure 3: E_{15}-page of the homotopy fixed point spectral sequence for $E_3^{hC_2}$.

G contains C_2, then we can "norm up" the Real bordism spectrum to G. This is an important construction, so we give it a name:

$$MU^{((G))} = N_{C_2}^G MU_{\mathbf{R}}$$

For formal reasons, we obtain a kind of "G-equivariant orientation" for Lubin–Tate spectra.

Theorem 22 (Hahn–S.)**.** *Let $G \leqslant \mathbf{G}(\kappa, \Gamma_n)$ be a finite subgroup containing the central subgroup C_2. Then there is a G-equivariant map*

$$MU^{((G))} \to E_n.$$

The motivation for doing this comes from Hill, Hopkins, and Ravenel's solution to the Kervaire invariant one problem. We have mentioned these three names and this problem quite a lot, so let us explore it now.

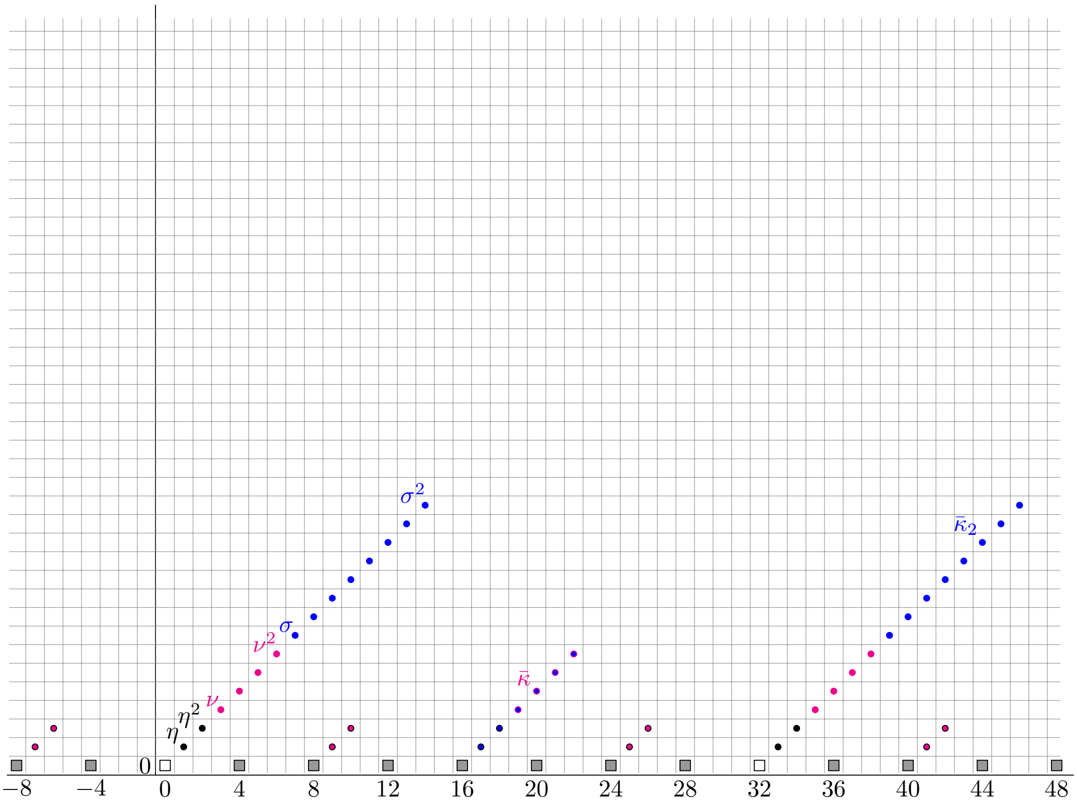

Figure 4: E_∞-page of the homotopy fixed point spectral sequence for $E_3^{hC_2}$.

5 Kervaire invariant one problem

Given a framed $(4k+2)$-dimensional manifold M, then Kervaire constructed a quadratic form

$$\phi\colon H^{2k+1}(M;\mathbf{Z}/2\mathbf{Z}) \to \mathbf{Z}/2\mathbf{Z}, \qquad \phi(x+y) = \phi(x) + \phi(y) + \langle x, y\rangle.$$

The *Kervaire invariant* $\Phi(M)$ of M is defined by the Arf invariant of ϕ. This is a fundamental invariant in differential and algebraic topology. Let us see how this invariant plays into questions about smooth structures on spheres and other manifolds. We say that a closed n-manifold Σ^n is a *homotopy n-sphere* if it is homotopy equivalent to S^n. The Generalised Poincaré conjecture asks:

Are all homotopy n-spheres Σ^n homeomorphic to S^n?

The answer is yes. For $n \geqslant 5$ this was proven by Smale in 1962, for $n = 4$ by Freedman in 1982, and for $n = 3$ by Perelman in 2002 (the cases for $n = 0, 1, 2$ are easy). Another natural question then relates not just the homotopical structure of a sphere to its topological type, but to its diffeomorphism type:

Are all homotopy n-spheres Σ^n diffeomorphic to S^n, equipped with the usual smooth structure?

For $n = 3$ this is true by Moise's theorem from 1952, but for $n = 4$ we still have no idea to this day. For $n = 7$, Milnor found counter examples in 1956 by constructing *exotic* 7*-spheres*, which are homotopy 7-spheres which are not diffeomorphic to the standard smooth structure on S^7. In 1963, Kervaire–Milnor computed the groups of possible exotic n-spheres (for $n > 4$) in terms of the stable homotopy groups of spheres $\pi_n^{\mathrm{st}} S^0$, modulo the Kervaire invariant. Let's state this result in some more detail. Write Θ_n for the group of homotopy n-spheres up to diffeomorphism (the group structure is given by connected sum), and Θ_n^{bp} for the group of homotopy n-sphere that bound parallelizable manifolds. The following is from [KM63].

Theorem 23 (Kervaire–Milnor)**.** *Fix $n \geqslant 5$. The subgroup Θ_n^{bp} is cyclic,*

$$|\Theta_n^{\mathrm{bp}}| = \begin{cases} 1 & n = 2k \\ 1 \text{ or } 2 & n = 4k+1 \\ b_k & n = 4k-1, \end{cases}$$

where $b_k = 2^{2k-2}(2^{2k-1} - 1) \cdot N(\frac{4B_{2k}}{k})$, and $N(-)$ indicates taking numerators, and B_{2k} Bernoulli numbers. The indecision in the $n = 4k+1$ case is due to the Kervaire invariant. There are also two exact sequences: the first for $n \not\equiv 2$ modulo 4

$$0 \to \Theta_n^{\mathrm{bp}} \to \Theta_n \to \pi_n^{\mathrm{st}} S^0/J \to 0.$$

and the second for $n \equiv 2$ modulo 4

$$0 \to \Theta_n^{\mathrm{bp}} \to \Theta_n \to \pi_n^{\mathrm{st}} S^0/J \xrightarrow{\Phi_n} \mathbf{Z}/2\mathbf{Z} \to \Theta_{n-1}^{\mathrm{bp}} \to 0. \tag{24}$$

Above, J indicates the image of the J-homomorphism and Φ_n the Kervaire invariant.

It is in this sense that understanding the Kervaire invariant Φ_n from (24) is the last missing piece of this Kervaire–Milnor puzzle. This leads us to the following question:

In which dimensions is there a framed manifold with Kervaire invariant one?

In other words, for which n is Φ_n from the second exact sequence of Theorem 23 nontrivial? In 1963, not much was known. It was know that there were such manifolds in dimensions 2, 6, and 14, and one could show there were no such manifolds in dimensions 10 or 18. There was no general structure theorems though, nor a direct connection to stable homotopy theory (apart from rewriting framed cobordism groups as stable homotopy groups of spheres). This was until the following theorem of Browder from 1969; see [Bro69].

Theorem 25 (Browder)**.** *If $\Phi(M) = 1$, then M has dimension $2^{j+1}-2$. Moreover, there exists a framed manifold of Kervaire invariant one if and only if an element $h_j^2 \in \mathrm{Ext}_{\mathcal{A}}^{2,2^{j+1}}(\mathbf{F}_2, \mathbf{F}_2)$ on the E_2-page of the mod 2 Adams spectral sequence for S^0 survives to an element $\theta_j \in \pi_{2^{j+1}-2} S^0$.*

Thus the question is now geared for homotopy theory, and one surrounding the existence or fate of these θ_j's. There are some approachable examples now:

- For $j = 1, 2, 3$, then the elements $h_j \in \mathrm{Ext}_{\mathcal{A}}^{1,2^j}(\mathbf{F}_2, \mathbf{F}_2)$ survive the Adams spectral sequence, as they detect the Hopf elements η, ν, and σ inside $\pi_* S^0$. This implies that θ_1, θ_2, and θ_3 also exist.
- Mahowald and Tangora (together with Barratt) show that $\theta_4 \in \pi_{30} S^0$ exists.
- Barratt, Jones, and Mahowald show that $\theta_5 \in \pi_{62} S^0$ exists.

However, the Adams spectral sequence becomes exceptionally hard to work with as the degrees grow, so we cannot continue in this direction. The question remains:

What is the fate of the higher θ_j's?

The solution to this problem results found in *ibid.*

Theorem 26 (Hill–Hopkins–Ravenel)**.** *For $j \geqslant 7$, the elements θ_j do not exist.*

Let us remark that the case of $\theta_6 \in \pi_{126} S^0$ is still open. The outline for the proof of the above theorem is rather simple, given other results due to Hill, Hopkins, and Ravenel.

Sketch of a proof. Start with $MU^{((C_8))}$. We then find a particularly nice class $D \in \pi_\bigstar^{C_8} MU^{((C_8))}$, then consider the spectrum formed by inverting that element $D^{-1}MU^{((C_8))}$, and set Ω to be the C_8-fixed points of this. There are now the following three theorems from [HHR16]:

- Detection theorem: If θ_j exists, then its image in $\pi_{2^{j+1}-2}\Omega$ is nonzero.
- Periodicity theorem: The spectrum Ω is 256-periodic.
- Gap theorem: For $i = -1, -2, -3$, the groups $\pi_i \Omega$ vanish.

Combing these three theorems together, one quickly sees that for $j \geqslant 7$, the elements θ_j cannot exists, as their image in Ω necessarily vanishes. □

Figure 5 displays a *slice spectral sequence*[12] for a baby form of Ω, which illustrates how one can visualise the gap and periodicity theorems in this simple case. In this example the spectrum in question has period 32, as opposed to Ω's period of 256.

In the proof of Theorem 26, Hill, Hopkins, and Ravenel use norms of $MU_{\mathbf{R}}$, but this was not the original plan. Indeed, these three authors showed that $E_4^{hC_8}$ also detects θ_j, however the resulting homotopy fixed point spectral sequence is so difficult that even a correct proof might not be understood by a majority of the community. This is the reason why they settled with using $MU^{((C_8))}$. Indeed, the pros of $MU^{((G))}$ is that it is a genuine equivariant homotopy type, which provides some ridigity to computations with the slice spectral sequence – a very useful feature. On the other hand, E_n is a Borel equivariant homotopy type (meaning we consider it as a cofree genuine G-spectrum), which is what makes computations so difficult. However, these Lubin–Tate spectra are perfect for doing chromatic homotopy theory. The Real orientation of E_n eliminates the above cons for Lubin–Tate spectra whilst retaining all the pros!

[12]The *slice spectral sequence* is a new kind of spectral sequence developed in [HHR16] for studying equivariant spectra.

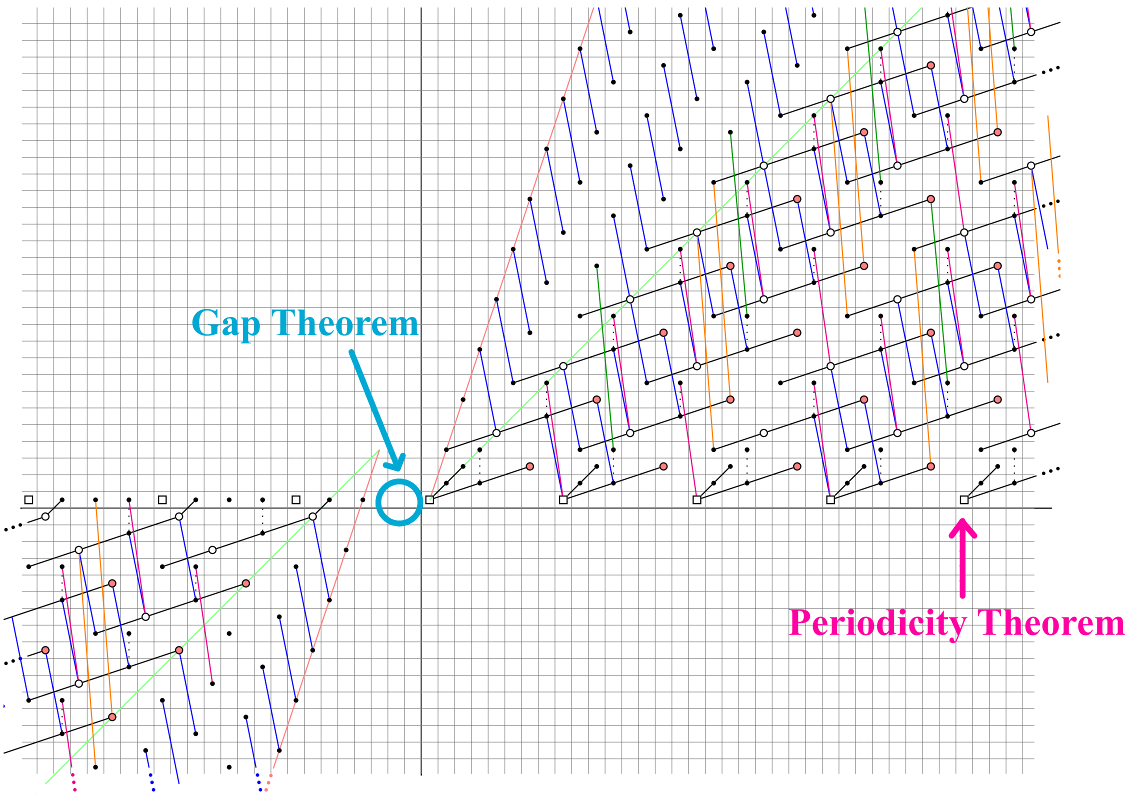

Figure 5: Slice spectral sequence for a baby form of Ω.

6 Detection tower

Combining equivariant homotopy theory with chromatic homotopy theory has led to plenty of recent research. One example of such is the new program to study the following *detection* *tower*, as formulated by Hill:

$$
\begin{tikzcd}
 & (MU^{((C_{2^m}))})^{C_{2^m}} \arrow[r] \arrow[d] & E_{2^{m-1}n}^{hC_{2^m}} \\
 & \vdots \arrow[d] & \\
S^0 \arrow[uur] \arrow[r] \arrow[dr] \arrow[ddr] & (MU^{((C_8))})^{C_8} \arrow[r] \arrow[d] & E_{4n}^{hC_8} \\
 & (MU^{((C_4))})^{C_4} \arrow[r] \arrow[d] & E_{2n}^{hC_2} \\
 & (MU_{\mathbf{R}})^{C_2} \arrow[r] & E_n^{hC_2}
\end{tikzcd}
$$

As we move up the tower, more elements in the Hurewicz image are detected and the theories become more and more powerful. One goal is to analyse this tower as much as possible. Before we outline the recent study of this detection tower, let us first recall the classical method one can use truncated forms of BP to model Lubin–Tate spectra. There is the following schematic diagram of spectra:

$$
\begin{tikzcd}
BP \arrow[r] & \cdots \arrow[r] & BP\langle 3\rangle \arrow[r] \arrow[d] & BP\langle 2\rangle \arrow[r] \arrow[d] & BP\langle 1\rangle \arrow[d] \\
 & & v_3^{-1}BP\langle 3\rangle \arrow[d, dotted] & v_2^{-1}BP\langle 2\rangle \arrow[d, dotted] & v_1^{-1}BP\langle 1\rangle \arrow[d, dotted] \\
 & & E_3 & E_2 & E_1
\end{tikzcd}
$$

When we p-localise MU, then this spectrum splits into copies of BP, the *Brown–Peterson spectrum*, a smaller spectrum than MU which also retains much of the chromatic information. These simpler p-local spectra BP are often better for calculations. For example, we have $\pi_* BP \simeq \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$, where $|v_n| = 2(p^n - 1)$. The truncations $BP\langle n\rangle$ have homotopy groups $\mathbf{Z}_{(p)}[v_1, \ldots, v_n]$. The formal group laws associated with $BP\langle n\rangle$ give us models for E_n. In the equivariant world there is *Real Brown–Peterson spectrum* $BP_{\mathbf{R}}$, which comes from a splitting of the 2-localisation of $MU_{\mathbf{R}}$, and by taking norms and we obtain an equivariant form of the classical case above:

$$\begin{tikzcd}
BP^{((C_{2^m}))} \arrow[r] & \cdots \arrow[r] & BP^{((C_{2^m}))}\langle 3\rangle \arrow[r] \arrow[d] & BP^{((C_{2^m}))}\langle 2\rangle \arrow[r] \arrow[d] & BP^{((C_{2^m}))}\langle 1\rangle \arrow[d] \\
 & & D_3^{-1}BP^{((C_{2^m}))}\langle 3\rangle \arrow[d, dotted] & D_2^{-1}BP^{((C_{2^m}))}\langle 2\rangle \arrow[d, dotted] & D_1^{-1}BP^{((C_{2^m}))}\langle 1\rangle \arrow[d, dotted] \\
 & & E_{3\cdot 2^{m-1}} \arrow[loop below, "C_{2^m}"] & E_{2\cdot 2^{m-1}} \arrow[loop below, "C_{2^m}"] & E_{2^{m-1}} \arrow[loop below, "C_{2^m}"]
\end{tikzcd}$$

The loops on the bottom of the diagram indicate that we are considering these Lubin–Tate spectra with C_{2^m}-actions. In some sense, these equivariant refinements also provide us with good models for equivariant Lubin–Tate spectra. The following contains some of the content of [BHSZ20].

Theorem 27 (Beaudry–Hill–S.–Zeng)**.** *The equivariant formal group laws associated with $BP^{((C_{2^m}))}\langle n\rangle$ give good models for $E_{2^{m-1}\cdot n}$ equipped with a C_{2^m}-action.*

Of course, it is subjective what the adjective "good" means above, however, this is somewhat justified by the computational and conceptual power of these models. Let us consider the case of $m = 2$:

$$\begin{tikzcd}
BP^{((C_4))} \arrow[r] & \cdots \arrow[r] & BP^{((C_4))}\langle 3\rangle \arrow[r] \arrow[d] & BP^{((C_4))}\langle 2\rangle \arrow[r] \arrow[d] & BP^{((C_4))}\langle 1\rangle \arrow[d] \\
 & & D_3^{-1}BP^{((C_4))}\langle 3\rangle \arrow[d, dotted] & D_2^{-1}BP^{((C_4))}\langle 2\rangle \arrow[d, dotted] & D_1^{-1}BP^{((C_4))}\langle 1\rangle \arrow[d, dotted] \\
 & & E_6 \arrow[loop below, "C_4"] & E_4 \arrow[loop below, "C_4"] & E_2 \arrow[loop below, "C_4"]
\end{tikzcd} \tag{28}$$

Starting with $BP^{((C_4))}\langle 1\rangle$, we see this spectrum is closely related to $\mathrm{TMF}_0(5)$ as studied by Behrens–Ormsby, Hill–Hopkins–Ravenel, and Beaudry–Bobkova–Hill–Stojanoska. We can actually calculate the slice spectral sequence (SliceSS) for this $BP^{((C_4))}\langle 1\rangle$; see Figure 6 and Figure 7.

Let us also note that the differentials for the homotopy fixed point spectral sequence for $E_2^{hC_4}$ come from the differentials for the slice spectral sequence for $BP^{((C_4))}\langle 1\rangle$; compare Figure 8 with Figure 9. This example shows how equivariant truncated Brown–Peterson spectra can help calculate the homotopy fixed points of Lubin–Tate spectra, now for groups larger than C_2.

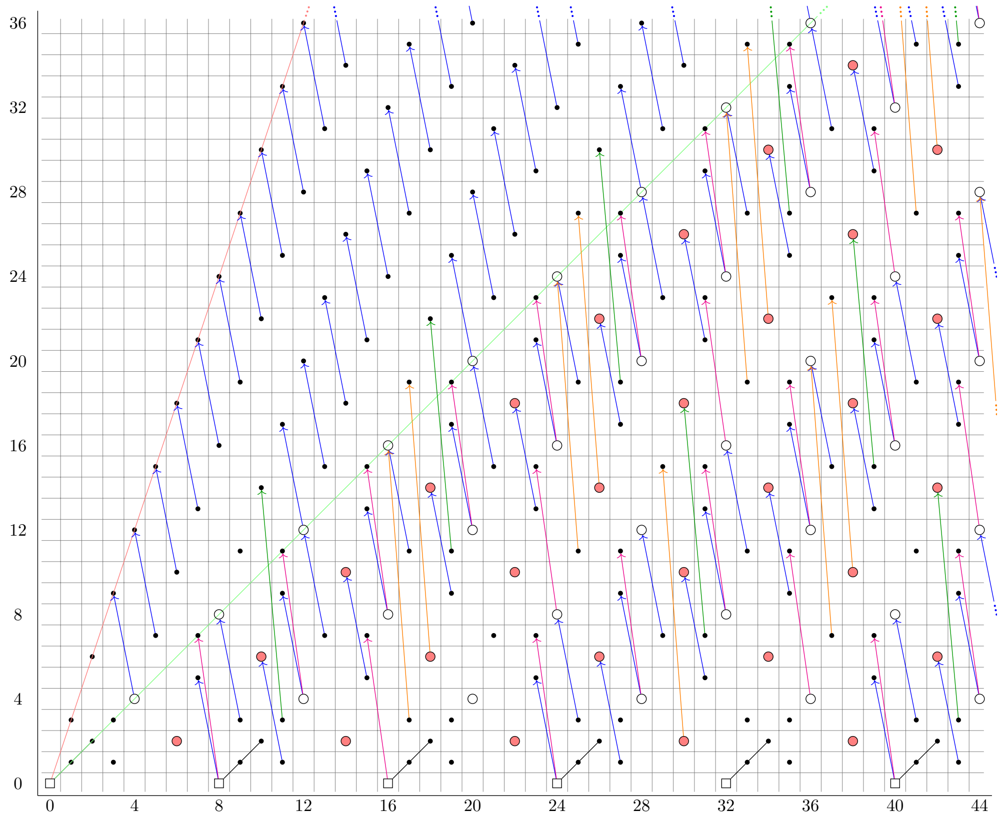

Figure 6: Slice spectral sequence for $BP^{((C_4))}\langle 1\rangle$ with differentials.

The next column in (28) is this spectrum $BP^{((C_4))}\langle 2\rangle$. Computations with this spectrum are also possible, due to work by Hill, S., Wang, and Xu; see [HSWX19]. In particular, this is the first example of a computation of the homotopy fixed points of Lubin–Tate spectra for heights greater than 2 and where the group is bigger than C_2. The slice spectral sequence for $BP^{((C_4))}\langle 2\rangle$ illustrates how much information there is to control; see Figure 10 and Figure 11.

One can also read the periodicity of $E_4^{hC_4}$ from this computation, it has period 128. This also implies that $E_4^{hC_{12}}$ has period 384. We can ask what the period of E_n^{hG} is for any finite group $G \leqslant \mathbf{G}_n$. This is answered by the following *periodicity theorem*.

Theorem 29 (Beaudry–Hill–S.–Wang–Xu–Zeng)**.** *The spectrum $E_{n\cdot 2^{m-1}}^{hC_{2^m}}$ is periodic with period $2^{n\cdot 2^{m-1}+m+1}$. The spectrum $E_{4n+2}^{hQ_8}$ is periodic with period 2^{4n+6}.*

This resolves the periodicity of E_n^{hG} at all heights and all G (at $p = 2$).

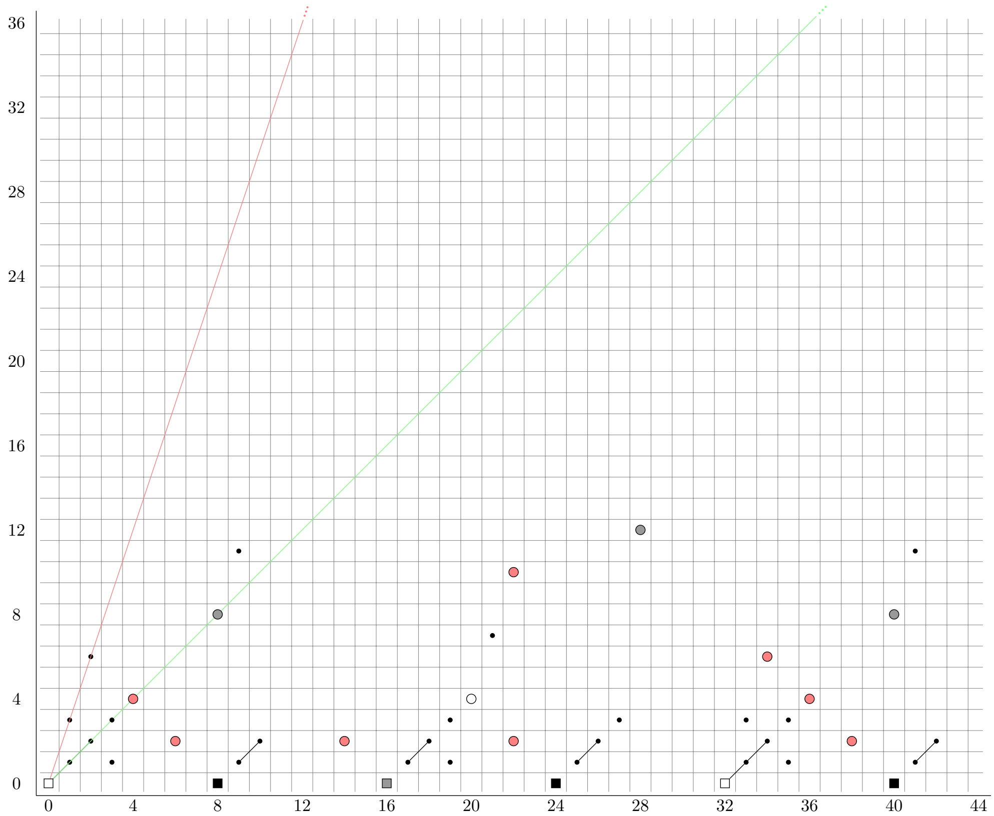

Figure 7: E_∞-page of the slice spectral sequence for $BP^{((C_4))}\langle 1\rangle$.

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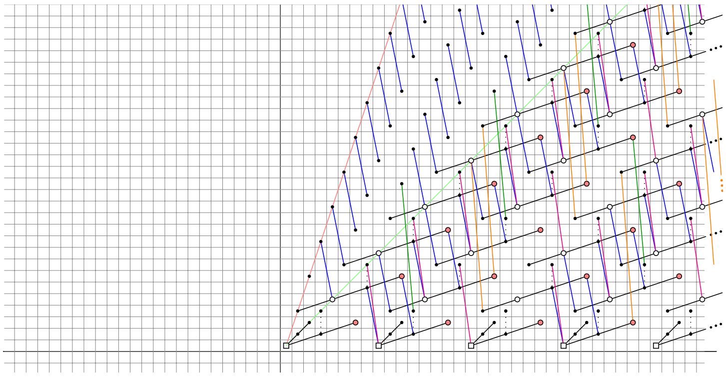

Figure 8: Reindexing of the slice spectral sequence for $BP^{((C_4))}\langle 1\rangle$ with differentials.

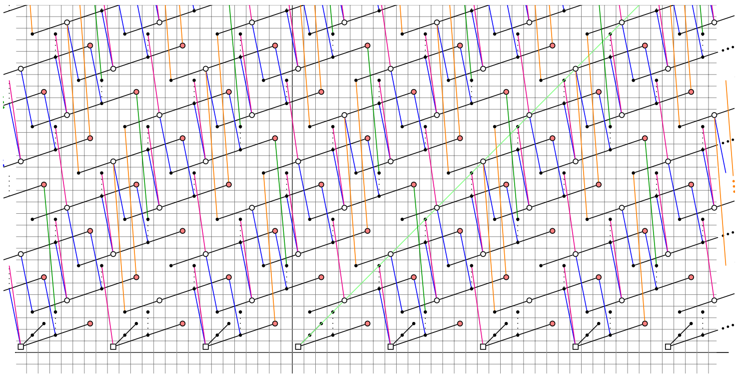

Figure 9: Homotopy fixed point spectral sequence for $E_2^{hC_4}$ with differentials.

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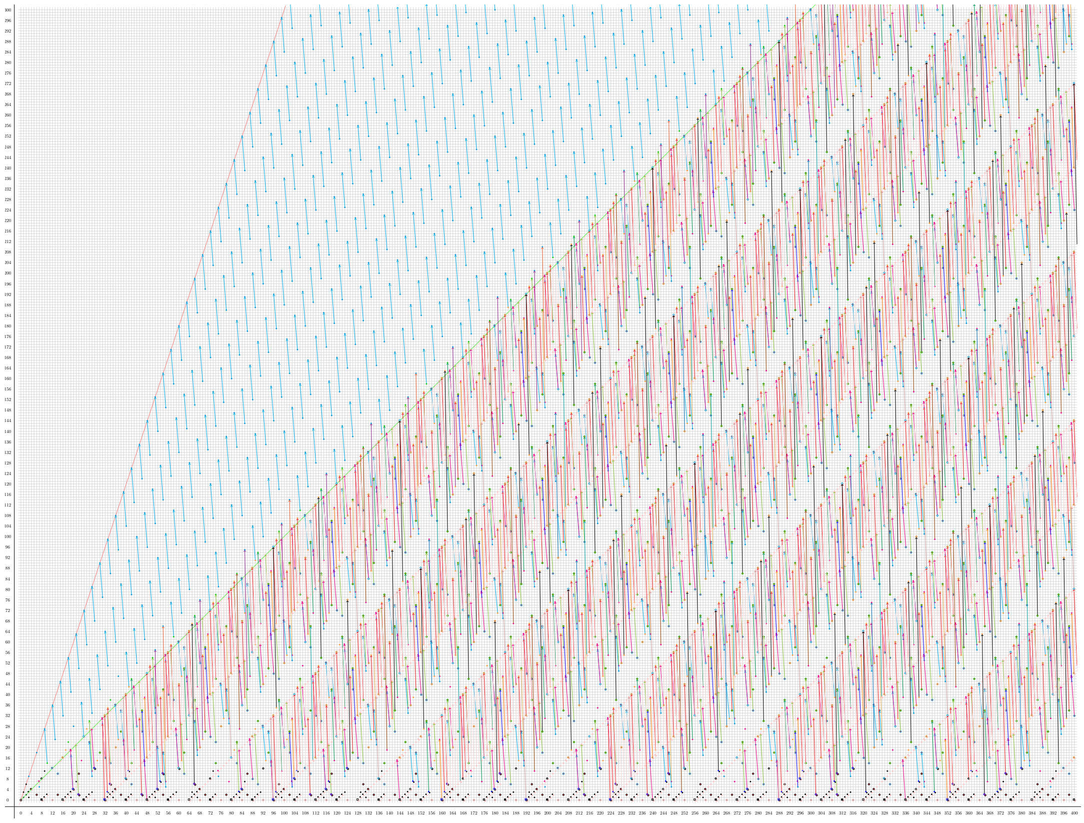

Figure 10: Slice spectral sequence for $BP^{((C_4))}\langle 2\rangle$ with differentials.

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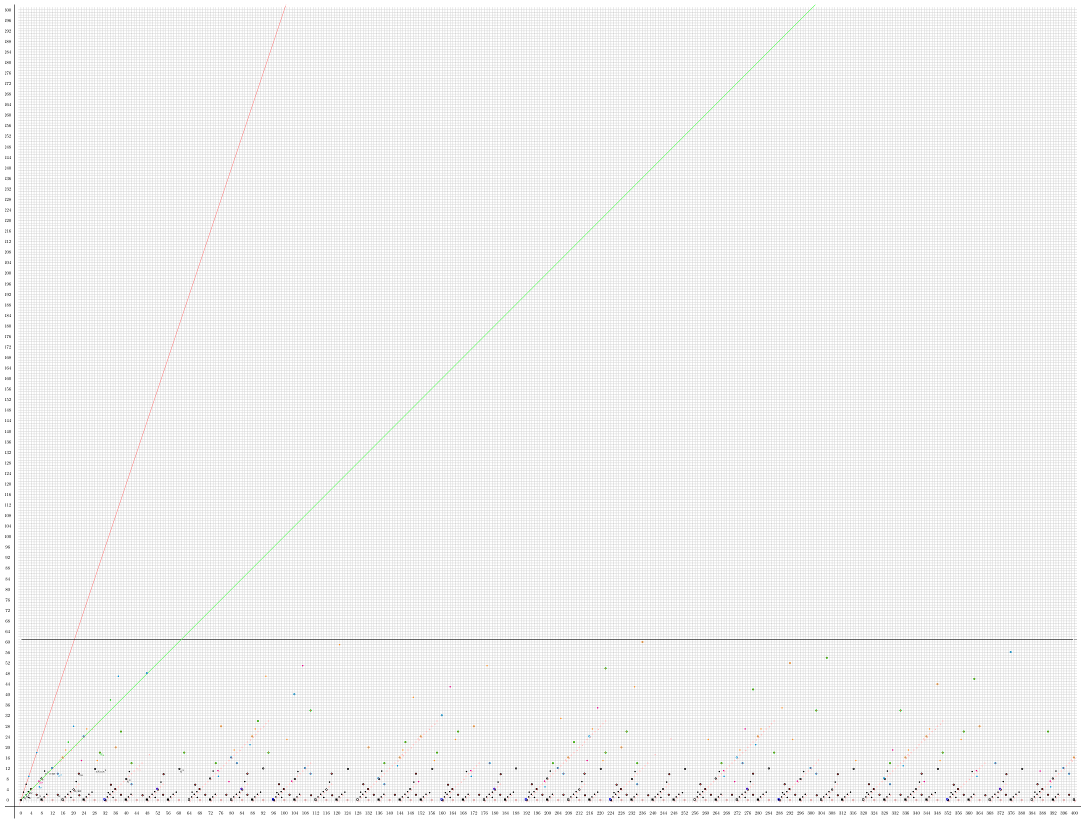

Figure 11: E_∞-page of the slice spectral sequence for $BP^{((C_4))}\langle 2\rangle$.

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