TALK 3: FINITENESS PROPERTIES OF MODULI SPACES OF HIGH-DIMENSIONAL MANIFOLDS

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1. Moduli space

Let M be a smooth manifold of dimension d.

Task: classify smooth M-bundles over a CW complex B.

Example 1. $M = \mathbb{R}^d$, the forgetful map induces an isomorphism

$$\{\text{vector bundles of rank } d \text{ over } B\}/\text{iso} \xrightarrow{\cong} \{\text{smooth } \mathbb{R}^d\text{-bundles over } B\}/\text{iso}.$$

One can construct a space m_d such that there is a bijection

$$(*) \qquad [B, m_d] \xrightarrow{\cong} \{\text{vector bundles of rank } d \text{ over } B\}/\text{iso}.$$

In fact, let

$$\begin{aligned} m_{d,k} &= \{V \subset \mathbb{R}^{d+k} | V \text{ is a linear subspace isomorphic to } \mathbb{R}^d\} \\ &\cong \operatorname{inj}(\mathbb{R}^d, \mathbb{R}^{d+k}) / \operatorname{GL}_d(\mathbb{R}), \\ m_d &= \operatorname*{colim}_k m_{d,k}, \end{aligned}$$

where inj is the subspace of linear maps with trivial kernel. Then, m_d satisfies the property (*).

In general, one can make the following definition

$$\begin{aligned} m_{M,k} &= \{W \subset \mathbb{R}^{d+k} | W \text{ is a diffeomorphic to } M\} \\ &\cong \operatorname{Emb}(M, \mathbb{R}^{d+k}) / \operatorname{Diff}(M), \\ m_M &= \operatorname*{colim}_k m_{M,k} \simeq \operatorname{B}\operatorname{Diff}(M). \end{aligned}$$

m_M is the moduli space of smooth manifolds diffeomorphic to M, in the sense that there is a bijection

$$[B, m_M] \xrightarrow{\cong} \{\text{smooth } M\text{-bundles over } B\}/\text{iso}.$$

Question: homotopy type of $\operatorname{B}\operatorname{Diff}(M)$, or H_*, π_* of it.
$H^*(\operatorname{B}\operatorname{Diff}(M))$, ring of characteristic classes of smooth M-bundles;
$\pi_*(\operatorname{B}\operatorname{Diff}(M))$, classifies M-bundles over spheres.

Recall that

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[2]Notes taken by Songqi Han.

- $H^*(m_d)$ is a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/2$. In particular, it's finitely generated. The corollary of this includes the computation of the bordism group and the signature theorem.
- $\pi_*(m_d)$ can only take values among the following, 0, $\mathbb{Z}/2$ and $\mathbb{Z}$. It is related to the classification of the exotic spheres.

Theorem 2 (B–Krannich–Kupers)**.** *Let M be a closed connected oriented smooth manifold with $\dim M = 2n \geq 6$. If $\pi_1 M$ is finite, then $\pi_* \operatorname{B}\operatorname{Diff}^+(M)$ and $H^{*-1}(\operatorname{B}\operatorname{Diff}^+(M))$ are finitely generated, for $* \geq 2$. $\operatorname{Diff}^+(M)$ stands for the oriented diffeomorphism group of M.*

Remark. When $* = 1$, $\pi_0 \operatorname{Diff}^+(M)$ is a group of type F_∞. This is proved by Sullivan for the case when $\pi_1 M$ is trivial, and generalised by Triantafillou to the case when $\pi_1 M$ is finite.

2. Low dimensional cases

dim = 2. Smale proved that

$$\operatorname{B}\operatorname{Diff}^+(S^2) \simeq \operatorname{B}\operatorname{SO}(3).$$

dim = 3. Hatcher proved that

$$\operatorname{B}\operatorname{Diff}^+(S^3) \simeq \operatorname{B}\operatorname{SO}(4).$$

In general, for every quotient space S^3/G of S^3 by isometric action, $\pi_*(\operatorname{B}\operatorname{Diff}(S^3/G))$ is finitely generated (Bamler and Kleiner proved the space of Riemannian metrics on S^3/G with constant sectional curvature 1 is contractible).

dim = 4. The theorem doesn't hold. Baraglia proved

$$\pi_2(\operatorname{B}\operatorname{Diff}(K3)) \supset \oplus_{i=1}^{\infty}\mathbb{Z}.$$

3. High dimension in 70-80s

Waldhausen proved there is a splitting up to homotopy of the A-theory space into the product of the stable homotopy theory and the smooth Whitehead space

$$A(M) \simeq Q_+(M) \times Wh^d(M).$$

The connected component of the loop space of the Whitehead space is homotopy equivalent to the classifying space of the stable pseudoisotopy group.

$$\Omega_0 Wh^d(M) \simeq \operatorname{B}\mathcal{P}(M) = \operatorname{colim}_k \operatorname{B} P(M \times D^k).$$

The pseudoisotopy group $P(M)$ of M is the subgroup of the diffeomorphism group of $M \times I$ fixing $\partial M \times I \cup M \times \{0\}$.

Igusa proved

$$\operatorname{B} P(M) \to \operatorname{B}\mathcal{P}(M)$$

is about $d/3$-connected. If the homotopy automorphism group (group of weak self homotopy equivalences) of M is known, then $\operatorname{B} P(M)$ can be related to $\operatorname{B}\operatorname{Diff}(M)$ by surgery theory.

Betley proved if $\pi_1 M$ is finite, then $\pi_* A(M)$ are finitely generated.

With these results, one can show for d large and $k < d/3$, $\pi_k \operatorname{B}\operatorname{Diff}(M)$ is finitely generated.

4. High dimension in 21st century

Proposition 3 (Galatius–Randal-Williams-Friedrich)**.** *Let M^{2n} be a compact manifold with finite π_1. Then*

(1) *$H_k(\operatorname{B}\operatorname{Diff}_\partial(M\#W_g))$ is independent of g. $W_g = S^n \times S^n$.*

(2) *stable homology is expressable in terms of a homotopy quotient of the connected component of the Madsen–Tillmann spectrum $\Omega_0^\infty MT\Theta$.*

As a corollary, $H_k(\operatorname{B}\operatorname{Diff}_\partial(M\#W_g))$ is finitely generated, for large g.

Proposition 4 (Goodwillie-Klein-Weiss)**.** *If M^d can be built from a point by attaching handles of index less than $d-2$, then for a manifold of dimension d,*

$$\operatorname{Emb}(M,N) = \lim(T_1 \leftarrow T_2 \leftarrow \cdots),$$

where

- *$T_1 = \operatorname{Bun}(TM, TN) \simeq \operatorname{Imm}(M,N)$ is the space of bundle maps,*
- *there is a fibration sequence $L_k \to T_k \to T_{k-1}$; L_k is the space of sections of a bundle over $\operatorname{Conf}_k(M)$, with F also related to the configuration space.*

If in addition, $\pi_1 M$ is finite, then $\pi_*(\operatorname{Emb}(M,M))$ is finitely generated.

Proposition 5 (Kupers-Weiss fibre sequence)**.** *If M is compact, there is a fibration sequence*

$$\operatorname{B}\operatorname{Diff}_\partial(\partial M \times I) \to \operatorname{B}\operatorname{Diff}(M) \to \operatorname{B}\operatorname{Emb}(M).$$

Since the left term is an E_1-space, i.e. a topological monoid, this fibration sequence can be extended to the right by one term $\operatorname{B}^2\operatorname{Diff}_\partial(\partial M \times I)$, which is simply connected.

Proof of the main theorem. (1) There is a fibration sequence

$$\operatorname{Diff}_\partial(K) \to \operatorname{Diff}(M) \to \operatorname{Emb}(K,M),$$

where K only contains handles of M of index ≤ 2. Since $\pi_* \operatorname{Emb}(K,M)$ is finitely generated, it suffices to prove $\pi_*(\operatorname{Diff}_\partial(K))$ is finitely generated.

(2) There is a fibration sequence by proposition 5

$$\operatorname{B}\operatorname{Diff}_\partial(\partial K \times I) \to \operatorname{B}\operatorname{Diff}_\partial(K) \to \operatorname{B}\operatorname{Emb}^{\cong}(K),$$

where $\operatorname{Emb}^{\cong}$ is the union of connected components of Emb of embeddings that are isotopic to diffeomorphisms. Since $\operatorname{B}\operatorname{Emb}^{\cong}(K)$ has finitely generated homotopy groups by proposition 4, it suffices to prove $\operatorname{B}\operatorname{Diff}_\partial(\partial K \times I)$ has finitely generated homotopy groups.

(3) Since $\operatorname{B}\operatorname{Diff}_\partial(\partial K \times I)$ is a connected topological monoid, it suffices to consider instead its classifying space $\operatorname{B}^2\operatorname{Diff}_\partial(\partial K \times I)$, which is simply connected. By a theorem of Serre, a simply connected space has finitely generated homotopy

groups if it has finitely generated homology groups. The problem reduces to prove $\mathrm{B}^2\,\mathrm{Diff}_\partial(\partial K \times I)$ has finitely generated homology groups.

(4) Observing that $\partial(K\#W_g) \cong \partial K$, it suffices to prove $\mathrm{B}^2\,\mathrm{Diff}_\partial(\partial(K\#W_g) \times I)$ has finitely generated homology groups.

(5) There is a fibration sequence by proposition 5

$$\mathrm{B}\,\mathrm{Diff}_\partial(K\#W_g) \to \mathrm{B}\,\mathrm{Emb}^{\cong}(K\#W_g) \to \mathrm{B}^2\,\mathrm{Diff}_\partial(\partial(K\#W_g) \times I).$$

By proposition 3, the left term has finitely generated homology groups (major difficulty here, related to nonempty boundary). With a standard spectral sequence argument, it suffices to show $\mathrm{B}\,\mathrm{Emb}^{\cong}(K\#W_g)$ has finitely generated homology groups.

(6) By proposition 4, $\mathrm{B}\,\mathrm{Emb}^{\cong}(K\#W_g)$ has finitely generated homotopy groups. By proposition 5, there is a short exact sequence

$$* \to A \to \pi_1\,\mathrm{B}\,\mathrm{Diff}_\partial(K\#W_g) \to \pi_1\,\mathrm{B}\,\mathrm{Emb}^{\cong}(K\#W_g) \to *.$$

where A is a finitely generated abelian group. Since the middle term is of type F_∞ (same difficulty as above), the right term is of type F_∞. One concludes the theorem with the following lemma. □

Lemma 6. *Let X be path connected. Suppose $\pi_1 X$ is of type F_∞ and $\pi_k X$ is finitely generated for every $k \geq 2$, then $H_*(X)$ is finitely generated.*

References

[1] Mauricio Bustamante, Manuel Krannich, and Alexander Kupers, *Finiteness properties of automorphism spaces of manifolds with finite fundamental group*, 2021.