Operads and configuration spaces

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Viva Talbot! – June 4th

Embedding calculus

(Part 1)

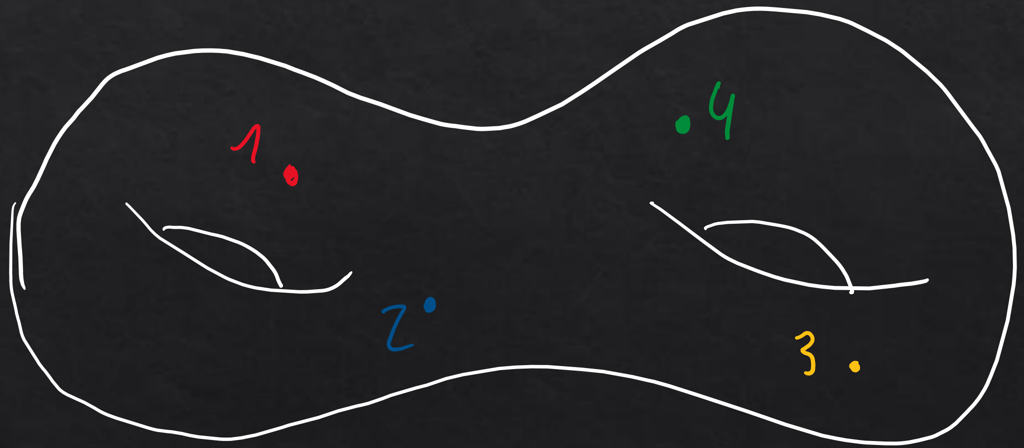
Motivation: $\mathrm{Emb}(M, N)$

- $\mathrm{Emb}(M, N) = \{\, f: M \hookrightarrow N \mid f \text{ is an embedding} \,\}$.
- Knot theory $= \pi_0 \mathrm{Emb}(\mathbb{S}^1, \mathbb{S}^3) \Rightarrow$ hard!
- When $\dim N - \dim M \geq 3$, $\pi_0 = \{*\}\ldots$
- …but higher π_k are interesting.
- Largest issue: $\mathrm{Emb}(-, N)$ is not "linear", i.e.,

$$\mathrm{Emb}(M \cup M', N) \neq \mathrm{Emb}(M, N) \times_{\mathrm{Emb}(M \cap M', N)} \mathrm{Emb}(M', N).$$

Configuration spaces

- The idea: approximate $\mathrm{Emb}(M, N)$ by maps between **configuration spaces**.
- $\mathrm{Conf}_M(r) \coloneqq \{(x_1, \dots, x_r) \in M^r \mid \forall i \neq j, x_i \neq x_j\}$.
- Classical objects in algebraic topology, initially for studying braids.



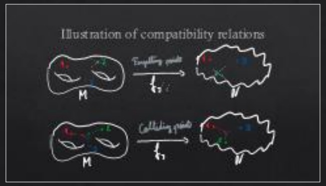
$\in \mathrm{Conf}_{\Sigma_2}(4)$

Goodwillie–Weiss manifold calculus

- $\mathrm{Emb}(M, N)$ is a subspace of:

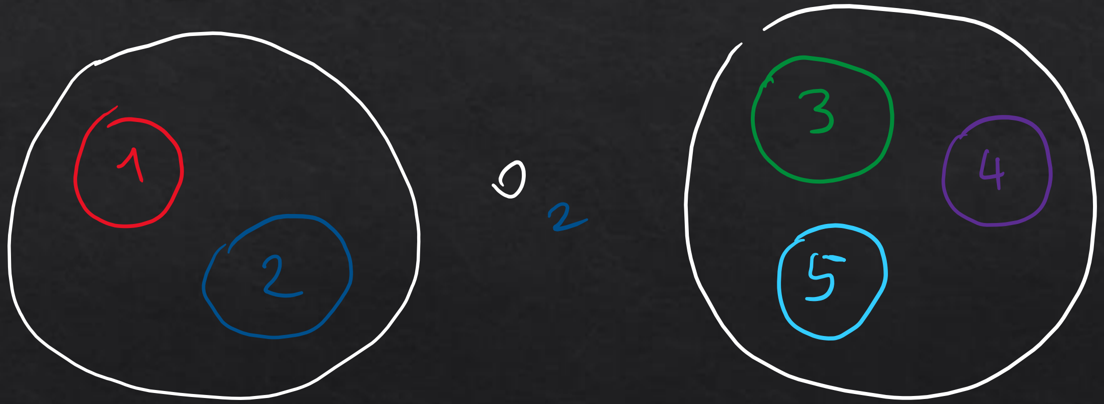
$$\mathrm{Map}_{\mathfrak{S}}(\mathrm{Conf}_M, \mathrm{Conf}_N) \coloneqq \prod_{r=0}^{+\infty} \mathrm{Map}_{\mathfrak{S}_r}(\mathrm{Conf}_M(r), \mathrm{Conf}_N(r))\,.$$

- Consider $\mathrm{Map}_{\mathfrak{S}}(\mathrm{Conf}_M^{\mathrm{fr}}, \mathrm{Conf}_N^{m-\mathrm{fr}})$ for unframed manifolds.
- $f \mapsto (f_r)_{r \geq 0}$ satisfies compatibility relations:
  - Forgetting in the source ↦ forgetting in the target;
  - Proximity in the source ↦ proximity in the target.
- ⇒ these relations are **relaxed "up to homotopy"**.



Operadic structure

- We want to clarify what "up to homotopy" means…
- $\Rightarrow$ we need operads!
- Let $\mathrm{D}_M^{\mathrm{fr}}(r) := \mathrm{Emb}(\coprod_{i=1}^{r} \mathbb{D}^m, M)$ and $\mathrm{D}_m^{\mathrm{fr}}(\mathrm{r}) := \mathrm{Emb}(\coprod_{i=1}^{r} \mathbb{D}^m, \mathbb{D}^m)$
- $\mathrm{D}_m^{\mathrm{fr}} := \{\mathrm{D}_m^{\mathrm{fr}}(r)\}_{r\geq 0}$ is the (framed) **little disks operad**:



Extra structure:
$\mathrm{D}_m^{\mathrm{fr}}(r) \times \mathrm{D}_m^{\mathrm{fr}}(s) \to \mathrm{D}_m^{\mathrm{fr}}(\mathrm{r}+\mathrm{s}-1)$

- $\mathrm{D}_M^{\mathrm{fr}} := \{\mathrm{D}_M^{\mathrm{fr}}(r)\}_{r\geq 0}$ is a **right module** over $\mathrm{D}_m^{\mathrm{fr}}$ via $\mathrm{D}_M^{\mathrm{fr}}(r) \times \mathrm{D}_m^{\mathrm{fr}}(s) \to \mathrm{D}_M^{\mathrm{fr}}(\mathrm{r}+\mathrm{s}-1)$

Operads & GW calculus

- Any embedding $f : M \hookrightarrow N$ induces a **morphism** $\mathrm{D}_M^{\mathrm{fr}} \to \mathrm{D}_N^{\mathrm{fr}}$, not just a map between configuration spaces!
- **Theorem** [Goodwillie–Weiss, Arone–Turchin, Turchin, Boavida–Weiss, Sinha…]. If $\dim N - \dim M \geq 3$, then
$$\mathrm{Emb}(M, N) \simeq \mathbb{R}\mathrm{Map}_{\mathrm{D}_m^{\mathrm{fr}}}\left(\mathrm{D}_M^{\mathrm{fr}}, \mathrm{D}_N^{m-\mathrm{fr}}\right).$$
- Upshot: if we know the homotopy type of the **collection** of configuration spaces **as right modules**, then we can compute embedding spaces.
- Computing $\mathrm{Conf}_M(r)$ is difficult. For example,
$$M \simeq M' \not\Rightarrow \mathrm{Conf}_M(r) \simeq \mathrm{Conf}_{M'}(r).$$

Bonus: deloopings

- Operads were initially introduced to study **iterated loop spaces**:
$$\Omega^n X \coloneqq \{\gamma \colon \mathbb{D}^n \to X \mid \gamma(0) = \gamma(1) = x_0\}.$$
- **Theorem** [Boardman–Vogt, May] If Y is an "algebra" over D_m, then $Y \simeq \Omega^m X$ for some X.
- The space $\mathrm{Emb}_\partial(D^m, D^n)$ is an algebra over D_m:

- $\Rightarrow \mathrm{Emb}_\partial(D^m, D^n) \simeq \Omega^m X$ where X has an operadic description [Dwyer-Hess, Arone-Turchin, Ducoulombier-Turchin]. (In fact, $\simeq \Omega^{m+1}$!)

Homotopy of configuration spaces

(Part 2)

Rational homotopy theory

- The whole homotopy type is too complex.
- We focus on **characteristic zero**.
- Definition: $f : X \to Y$ is a **rational equivalence** if
  $$\pi_*(f) \otimes_{\mathbb{Z}} \mathbb{Q} : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$ is an isomorphism.
- Theorem [Sullivan]: There is an equivalence $\Omega^* \dashv \langle - \rangle$ between:
  - Simply connected finite-type spaces, up to rational equivalence;
  - Simply connected finite-type commutative differential-graded algebras, up to quasi-isomorphism.
- Upshot: we want to find a **model** of $\Omega^*(\mathrm{D}_M^{\mathrm{fr}})$ with its action of $\Omega^*(\mathrm{D}_m^{\mathrm{fr}})$.

Building brick: $\mathbb{R}^m$

- The cohomology $H^*\big(\mathrm{Conf}_{\mathbb{R}^m}(r)\big) = H^*\big(\mathrm{D}_m(r)\big)$ is well-known [Arnold, Cohen]:

$$H^*(\mathrm{D}_m(r);\mathbb{Q}) = \frac{S(\omega_{ij})_{1\le i\neq j\le r}}{(\omega_{ij}^2 = \omega_{ji} - (-1)^m\omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0)}.$$

- **Theorem** [Kontsevich, Tamarkin, Lambrechts–Volić, …….] The operad D_m is formal, i.e.,

$$H^*(\mathrm{D}_m;\mathbb{Q}) \simeq \Omega^*(\mathrm{D}_m).$$

- Many important consequences, e.g., deformation quantization, Deligne conjecture…

Formality: two approaches

- Kontsevich's approach:
  - Replace the 3T relation by "internal vertices";
  - Prove combinatorically that we have a resolution of $H^*(\mathrm{D}_n)$;
  - Use integrals to connect with $\Omega^*(\mathrm{D}_n)$.
- [Giansiracusa–Salvatore] formality of $\mathrm{D}_2^{\mathrm{fr}}$.

- Tamarkin's approach:
  - Find a simpler groupoid $\mathrm{PaB} \simeq \pi\mathrm{D}_2$;
  - Find a (Koszul) resolution of $H^*(\mathrm{D}_2)$, the Drinfeld–Kohno Lie algebra;
  - Connect the two with a Drinfeld associator.
- [Ševera] Formality of $\mathrm{D}_2^{\mathrm{fr}}$.
- **Theorem** [CIW] Cyclic formality of $\mathrm{D}_2^{\mathrm{fr}}$. Proof inspired by Ševera's.

Configuration spaces of closed manifolds

- Model of $\mathrm{D}_M(r) \simeq \mathrm{Conf}_M(r) = M^r \setminus \bigcup_{i \neq j} \Delta_{ij}$ conjectured by Lambrechts–Stanley:
  - Generators: ω_{ij} for $1 \leq i \neq j \leq r$; x_i for $1 \leq i \leq r$ and $x \in A \simeq \Omega^*(M)$.
  - Relations:
    - Same as before: $\omega_{ij}^2 = \omega_{ji} - \omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0;$
    - Symmetry: $x_i\omega_{ij} = x_j\omega_{ij}$.
  - Differential: $d\omega_{ij} = \Delta_{ij}$.
- **Theorem** [I, cf. Campos-Willwacher] The above *is* a model for M simply connected; operadic structure if $\dim M \geq 4$.
- **Corollary.** Real homotopy invariance.

Configuration spaces of surfaces

Theorem [CIW]. Small, explicit model $\mathrm{G}^{\mathrm{fr}}_{\Sigma_g}$ of $\mathrm{D}^{\mathrm{fr}}_{\Sigma_g}$, in arity r:

- Generators: ω_{ij} for $1 \leq i \neq j \leq r$; $\alpha_{1,i}, \dots, \alpha_{g,i}, \beta_{1,i}, \dots, \beta_{g,i}$ for $1 \leq i \leq r$; θ_i for $1 \leq i \leq r$.
- Relations:
  - Same as before: $\omega_{ij}^2 = \omega_{ji} - \omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$;
  - $\alpha_{k,i}\beta_{k,i} = \alpha_{l,i}\beta_{l,i}$ (volume form of Σ_g) and 0 otherwise;
  - Symmetry: $\alpha_{k,i}\omega_{ij} = \alpha_{k,j}\omega_{ij}, \beta_{k,i}\omega_{ij} = \beta_{k,j}\omega_{ij}$, $\theta_i\omega_{ij} = \theta_j\omega_{ij}$.
- Differential: $d\omega_{ij} = \Delta_{ij}$ and $d\theta_i = (2-2g) \cdot \mathrm{vol}_i$.
- Proof: $\mathrm{G}^{\mathrm{fr}}_{\Sigma_g} \xleftarrow{\text{Combin.}} \mathrm{Graphs}^{\mathrm{fr}}_{\Sigma_g} \xrightarrow{\mathrm{K}} \mathrm{IterHoch}\left(H^*\left(\mathrm{D}^{\mathrm{fr}}_{S^2\setminus 2g}\right); H^*(\mathrm{D}^{\mathrm{fr}}_{S^1\times\mathbb{R}})\right) \xleftarrow{\mathrm{T}} \Omega^*\left(\mathrm{D}^{\mathrm{fr}}_{\Sigma_g}\right)$.

Thank you for your attention!

These slides, the paper: https://idrissi.eu