Galois Extensions in Chromatic Homotopy

Lior Yanovski

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- **Examples**: HR, $\mathbb{S}$, $\mathbb{S}[G]\,(=\Sigma^\infty_+ G)$, KU, KO, ...

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Arithmetic Pullback Square.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widehat{X}_p \arrow[d] \\ X[p^{-1}] \arrow[r] & (\widehat{X}_p)[p^{-1}]. \end{tikzcd}$$

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$$L_{T(n)}X := \widehat{X}_{(p,v_1,\dots,v_{n-1})}[v_n^{-1}] \quad \in \quad \mathrm{Sp}_{T(n)}.$$

Chromatic Picture

Prime ideals. A chain under specialization:

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Residue fields. Morava K-theories:

$$\begin{array}{ccccccccccccc} K(0) & , & K(1) & , & K(2) & , & \dots & , & K(n) & , & \dots & , & K(\infty) \\ \| & & \wr\wr & & & & & & & & & & \| \\ H\mathbb{Q} & & KU/p & & & & & & & & & & H\mathbb{F}_p \end{array}$$

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- $\mathrm{Sp}_{T(n)}$ – less computable, related to unstable homotopy, redshift in algebraic K-theory.

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Example. $KO \to KU$ is a $\mathbb{Z}/2$ Galois extension.

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Group algebra. For every $m \in \mathbb{N}$:

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Splitting. $R[C_{p^r}] \simeq R[C_{p^{r-1}}] \times R[\omega_{p^r}].$

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Example. The **spherical Witt vectors**

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Lift the $\mathbb{Z}/d$-Galois extension $\mathbb{Z}_p \to W(\mathbb{F}_{p^d})$.

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Generalization (Tyler Lawson). Adjoining roots of arbitrary elements $a \in \pi_0 R^\times$:

$$\pi_0(R[a^{1/n}]) \simeq \pi_0(R)[t]/(t^n - a)$$

via *twisted* group rings.

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Abelianization. We have $\det\colon S_n \to \mathbb{Z}_p^\times$ and $\mathbb{G}_n^{\mathrm{ab}} \simeq \widehat{\mathbb{Z}} \times \mathbb{Z}_p^\times$.

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Theorem (Devalapurkar)

For $n \geq 1$, there is no $K(n)$-local commutative ring spectrum R, such that $\pi_0(R)$ contains a primitive p-th root of unity.

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Primitivity. There is a natural definition for $\omega^{(n)}$ to be *primitive*.

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Theorem (Westerland)

The algebra $\mathbb{S}_{K(n)}[\omega_{p^r}^{(n)}]$ *is* $(\mathbb{Z}/p^r)^\times$*-Galois and is classified by*

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Slogan. "$\det\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character"

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∞**-Semiadditivity** (Hopkins-Lurie). For all π-finite spaces A,

$$\varinjlim_A \xrightarrow{\sim} \varprojlim_A.$$

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Fact. For $\mathrm{Sp}_{K(n)}$, the map $|B^n C_p|$ is invertible.

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Fiber sequence.

$$B^nC_p \to B^nC_{p^r} \to B^nC_{p^{r-1}}.$$

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Splitting.

$$\mathbb{S}_{K(n)}[B^nC_{p^r}][(1-\varepsilon)^{-1}] \simeq \mathbb{S}_{K(n)}[\omega_{p^r}^{(n)}].$$

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Corollary

We can lift every finite ***abelian*** *G-Galois extension $\mathbb{S}_{K(n)} \to R$, to a G-Galois extension $\mathbb{S}_{T(n)} \to R^f$.*

Thank You!