

# Avatars for algebraic K-theory in the motivic gather.town

Trend of the last years: interdisciplinary studies  
 Part of this trend: motivic homotopy theory  
 homotopy theory  $\curvearrowright$  algebraic geometry  
 methods objects

Our gather.town (homotopy!): spaces

More precisely, I mean the  $\infty$ -category  $\text{Spc}$ ,  
 whose objects can be modelled by sSet or CW-compl.

New perspective: SmMan are particularly nice,  
 we can gather around them

$$\text{Spc} \rightarrow \text{PSh}(\text{SmMan}) \quad X \mapsto \text{Hom}(-, X)$$

$$\begin{array}{ccc} \text{local} & \curvearrowleft & U \\ \text{contractibility} & \curvearrowright & \text{PSh}_{\text{MV}, \text{HT}}(\text{SmMan}) \\ F(M) \cong F(M_1) \times F(M_2) & \xrightarrow{\quad \quad \quad} & F(M \times [0, 1]) \cong F(M) \\ & \curvearrowleft & F(M_1 \cap M_2) \end{array}$$

$\text{SmMan}$  vs  $G$ - $\text{SmMan}$  <sup>(compact lie)</sup> given <sup>(+take care of infinite covers)</sup> genuine  $G$ -spaces.

Using this perspective, we can build the  
 Motivic gather.town (motivic-homotopy!)  
 $k$ -base field (could be any scheme)  
 $\text{SmMan}$  vs  $\text{Sm}_k$

$$\mathrm{PSh}(\mathrm{Sm}_k) \hookrightarrow \mathrm{PSh}(\mathrm{Sm}_k)^{\text{Nis, A'}} =: \mathrm{Spc}(k)$$

biased choice of descent  
so that alg. k-theory  
would be accepted (far < Nis < étale)

Examples: 1)  $\mathrm{PSh}(\mathrm{Sm}_k) \ni$  spaces, smooth k-schemes, stacks (restrictions to  $\mathrm{Sm}_k$ )  
2)  $\mathrm{Spc}(k) \ni \mathbb{G}_m : \mathrm{Spec} R \mapsto R^\times$

Betti realization:  $\mathrm{Spc}(\mathbb{C}) \rightarrow \mathrm{Spc}$   
 $\mathrm{Sm}_{\mathbb{C}} \ni X \mapsto X(\mathbb{C})$

Actually, why K-theory?.. an invariant  
built out of vector bundles.

In topology, rank n G-vector bundles  
on a (compact Hausd.) space U are given by  
 $\#_0 \mathrm{Maps}(U, \mathrm{Gr}_n^G)$   
 $\hookrightarrow \mathrm{Bl}_n$

However, alg geometry is more rigid:  
 there's a rank 2 v.b. on  $\mathbb{P}^1 \times \mathbb{A}^1$   
 whose fiber over 0 is  $\mathcal{O}^2$  and over 1 is  $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ .

Exercise!

So, vector bundles are only representable on affines:

Then (Asok-Kayser-Wendt), if  $U$  smooth affine  $k$ -scheme, then

$$\left\{ \text{rank } n \text{ v.b. } / \cong \right\} \simeq \underset{\text{Spc}(k)}{\text{Maps}} \left( U, \overset{\text{is } \text{GL}_n}{\text{Gr}_n} \right)$$

cool stuff! But only on affines...  
Asok-Fasli classification results for v.b.

But we want more inclusivity, so we'd like to have a vector bundle related invariant for non-affine smooth  $k$ -schemes as well.

adds freedom & complications  
{like denotacy}

- $\bullet K_0(\text{Spec } R) := (\text{vector bundles on } \text{Spec } R / \cong, \oplus)$  GP

Example: 1)  $K_0(\text{Spec } F) \simeq \mathbb{N}^{\text{gp}} = \mathbb{Z}$

2)  $K_0(C) \simeq \mathbb{Z} \times \text{Pic}(C)$  for  $C$  smooth curve

- To define higher  $K$ -theory, the idea is to remember the isoms instead of taking the quotient. Vect - stack of v.b.,  $\text{Vect}(X)$  for a scheme  $X$  is the groupoid of v.b. on  $X$ .

(CW-complex whose 0-cells are v.b. on  $X$  and 1-cells are isoms between v.b.)

$$K(\text{Spec } R) := \text{Vect}(\text{Spec } R)^{\text{gp}}$$

Thomason-Trobaugh  
 $K$ -theory space

gp is not an explicit operation:  
it's formally adding negative elements on  $\mathbb{H}_0$ ,  
but it creates chaos in higher homotopy groups.  
I had trouble understanding negative integers...  
more than with infinity!

On non-affine schemes:

$K: \text{Sm}_k^{\text{op}} \rightarrow \text{Spc}$  - Zariski sheaf of spaces s.t.  
for  $U$  affine,  $K(U) \cong \text{Vect}(U)^{\text{gp}}$ .

$K$ -theory groups:  $K_i(X) := \pi_i K(X)$ .

Nb!  $K(-)$  is a sheaf but  $K_i(-)$  aren't sheaves.

Prop.:  $K$ -theory space

$$K: \text{Sm}_k^{\text{op}} \rightarrow \text{Spc}$$

$$\in \text{Spc}(k)$$

After the break: various geometric avatars

(representability)!

Thm A (Morel-Voevodsky):  $X \in \text{Sm}_k \Rightarrow$   
 $K(X) \cong \text{Maps}_{\text{Spc}(k)}(X, \mathbb{Z} \times \text{Gr}_\infty)$   
 $\cong \text{Maps}_{\text{Spc}(k)}(X, \mathbb{Z} \times \text{Gr}_\infty)$   
 So,  $K \cong \mathbb{Z} \times \text{Gr}_\infty$  motivic avatar  
 $A^1$ -equivalence on affines of  $K$ -theory

This is nice because  $\text{Gr}_\infty$  has a stratification by affine spaces, so its homotopy type is easy.

$$B\mathbb{E}(k) \cong \mathbb{Z} \times BU = \Omega^\infty(kU)$$

Thm B (HINITY):  $X \in \text{Sm}_k \Rightarrow$

$$K(X) \cong \text{Maps}_{\text{Spc}(k)}(X, \mathbb{Z} \times \text{Hilb}_\infty(\mathbb{A}^\infty)),$$

different avatar

where  $\text{Hilb}_d(\mathbb{A}^n)(T) = \{ \mathcal{Z} \subset \mathbb{A}^n \times T \mid$   
 finite flat  $\xrightarrow{\text{deg } d} T \}$

(for  $T = \text{Spec } F$ ,

$\mathcal{Z}$  = a bunch of points, possibly with nilpotents)

$\text{Hilb}_d$  locus of smooth points ( $d$ -tuples)  $\approx \text{Conf}_d$

This is very unexpected, because  $\text{Hilb}_d(\mathbb{A}^n)$  has all possible singularities in the universe (Toadum)! Murphy's law

Actually, we proved a stronger statement:

$$\text{Gr}_{d-1} \xrightarrow{\cong} \text{Hilb}_d(\mathbb{A}^\infty)$$

$A^1$ -htpy equiv. on affines

Main idea: forget about embeddings into  $A^\infty$ ,  
 then we have (Whitney's embedding thm)

$$\text{Vect}_{d,-} \rightarrow \text{FFlat}_d$$

(over  $\text{Spec } R$ )  $V \mapsto R \oplus V$  square zero extension

$$A/R \hookleftarrow A$$

Claim  $A \hookrightarrow R \oplus A/R$  is  $A^1$ -htpic to  $\text{id}_{\text{FFlat}(R)}$ .

Explicit  $A^1$ -htpy is provided by:

$$\text{Rees}(A) := \{a_0 + a_1 t + \dots \mid a_i \in R\} \subset R[t]$$

$R[t]$ -algebra

$$\text{Rees}(A)/(t-1) \cong A; \quad \text{Rees}(A)/(t) \cong R \oplus A/R.$$

■

There are also variations of these results for other relatives of  $k$ .

Hermitian  $k$ -theory space:

$$GW(U) := \underset{\text{affine}}{\text{Vectsym}(U)^{\text{SP}}} \quad \text{v.b. with non-deg symm. bil. form}$$

$GW_0(\text{Spec } R)$  - Grothendieck-Witt group of  $R$

In general, define as a Zariski sheaf.

Their  $A^1$  (Schlichting-Tripathi).  $\text{char } k \neq 2 \Rightarrow$

$$GW(X) \cong \text{Maps}_{\text{Spe}(k)}(X, \mathbb{Z} \times \text{GrO}_\omega),$$

where  $\text{GrO}$  parametrizes subspaces where  $B(GW) = \mathbb{Z} \times BO \cong \mathbb{Z} \cong KO$   
 $H$  is non-degenerate.

Thm B' (KJNY).  $k$  any char  $\Rightarrow$

$$GW(X) \cong \text{Maps}_{\text{Spec}(k)}(X, \mathbb{Z} \times \text{Hilb}^{\text{Gor}, \text{or}}(\mathbb{A}^\infty)),$$

where  $\text{Hilb}^{\text{Gor}, \text{or}}$  parametrizes

Gorenstein closed subschemes, equipped  
with an orientation.

A finite flat  $R$ -algebra  $A$  is Gorenstein if its  
dualizing module  $\omega_{A/R} := \text{Hom}_R(A, R)$   
is invertible.

An orientation is a choice of a  
trivialization of  $\omega_{A/R}$  (neither  $\exists$  nor!).

WIP with Elden & Denis:

Thm  $A^u$  and  $B^u$  for K-theory of  
 $A$ -twisted vector bundles (at an Azumaya algebra).