

Invertible objects in stable homotopy theory

In algebra : R commutative ring



$$R^\times$$

units for
mult in R

$$H^0(\mathrm{Spec} R, \mathbb{G}_m)$$

$\mathbb{G}_m = \mathrm{Spec} \mathbb{Z}[t^{\pm 1}]$

$$\mathrm{Pic}(R)$$

units for
 \otimes_R in Mod_R

$$H^1(\mathrm{Spec} R, \mathbb{G}_m)$$

$$\mathrm{Br}(R)$$

~ units for
 \otimes_{Mod_R} in Cat_R

$$H^2(\mathrm{Spec} R, \mathbb{G}_m)$$

In stable homotopy theory:

R commutative ring spectrum

① Units of R

[May + Quinn-Ray-Tornehave
" ∞ -ring spaces and ∞ -ring spectra"]

$$\begin{array}{ccc} GL_1(R) & \xrightarrow{\text{map of spaces}} & \Omega^\infty R \\ \downarrow & & \downarrow \\ (\pi_0 R)^\times & \longrightarrow & \pi_0 R \end{array}$$

→ ∞ -loop space with \otimes , grouplike \Rightarrow spectrum $gl_1(R)$

$gl_1(R) \dashrightarrow R$ no map of spectra

Exercise

a) What are $\pi_* GL_1 R$?

b) For a space X , show $[X, GL_1 R] = R^0(X)^\times$

Vista: Rezk "the units of a ring spectrum
and a logarithmic cohomology
operation"

$$gl_1 R \xrightarrow{\log} L_{\underbrace{K(n)}_{\text{Morava K-theory}}} R$$

Algebra:

$$R^\times = \text{Maps}_{\text{rings}}(\mathbb{Z}[t^{\pm 1}], R)$$

$$\mathbb{G}_m(R) = \text{Map}_{\text{CatAlg}}(\text{Free comm. alg. on an f.w. gen.}, R)$$

(1)

Strict units of R

$$\mathbb{G}_m(R) := \text{Map}_{\text{CatAlg}}(\underbrace{\mathbb{Z}[[t^{\pm 1}]])}_{\mathbb{Z}_+^\infty}, R)$$

$$\simeq \text{Map}_{\text{Sp}}(H\mathbb{Z}, gl_1 R)$$

Exercise: Can you say anything about
 $\pi_K \mathbb{G}_m(R)$

② The Picard space / spectrum

$\text{Pic}(R)$

$$= \left\{ \begin{array}{l} \text{Space of invertible} \\ R\text{-modules \& equivalences} \end{array} \right. + \left. \bigotimes_R \right\}$$

[Ando-Blumberg-Gepner-Hopkins-Rozk
"Units of ring spectra and their spectra"]

$$\pi_0 \text{Pic}(R) = \frac{\text{invert. } R\text{-modules}}{\text{modulo equivalences}}$$

$$\Omega \text{Pic}(R) = \text{GL}_1(R)$$

$$(\text{Pic}(R))_R = \text{BGL}_1(R)$$

↑ conn. comp. of R

→ conn. spectrum $\text{pic } R$

$$[X, \text{Pic}(R)] = \text{Bundles of invertible } R\text{-modules on } X$$

$$\text{eg. } [X, \text{Pic}(S^0)] = \text{stable spherical fibrations on } X$$

Exercise: Invertible S^0 -modules?

Example: k_0 : conn. real K-theory

$\Omega^\infty k_0 \simeq \mathbb{Z} \times BO$ classifies stable ^{real} v. bundles

$$[X, \mathbb{Z} \times BO] \quad \xi/X$$

$$\downarrow$$

$$[X, \text{Pic } S^0] \quad \text{th}(\xi)$$

$$\rightsquigarrow \text{∞-loop map} \quad \mathbb{Z} \times BO \xrightarrow{k_0} \text{Pic } S^0$$

Adams' J -homomorphism!

Vista: orientations and Thom spectra

$$X \xrightarrow{f} \widetilde{\text{BGL}}(R)$$

$$\rightsquigarrow \text{colim}(X \xrightarrow{f} \widetilde{\text{BGL}}(R) \hookrightarrow \text{Mod}_R) = M_f$$

[May, ABGHR]

Vista: \rightarrow twisted R-cohomology

\rightarrow twists by ordinary coh. classes
 $\Rightarrow \text{Pic} = \text{Map}(H\mathbb{Z}, \text{pic})$

[Atiyah-Segal, Freed-Hopkins-Teleman, Ando-Blumberg-Gepner, Sati-Westerland, ...]

③ the Brauer Space

$\text{Br}(R)$

Baker-Richter-Szymik
Antieau-Gepner,
Gepner-Lawson,
Hopkins-Lurie

How to compute anything?

$$\pi_0 \text{Br}(R)$$

Brauer group

$$\pi_1 \text{Br}(R) = \pi_0 \text{Pic}(R)$$

Picard group

$$\pi_2 \text{Br}(R) \simeq \pi_1 \text{Pic}(R) = \pi_0 \text{GL}_1(R) = (\pi_0 R)^{\times}$$

$$\pi_{>2} \text{Br}(R) = \pi_3 \text{Pic}(R) = \pi_{>0} \text{GL}_1(R) = \pi_{>0} R$$

Example : the first k -invariant
of $\text{Pic}(R)$

$$\text{pic } R \longrightarrow \underbrace{H(\pi_0 \text{pic } R)}_{k \hookrightarrow \text{homomorphisms}} \longrightarrow \sum^2 H(\pi_1 \text{pic } R)$$

$$\begin{aligned} k &\hookrightarrow \text{homomorphisms} \\ \pi_0 \text{pic } R &\longrightarrow (\pi_1 \text{pic } R)[2] \\ &\quad (\pi_0 R)^{\times}[2] \end{aligned}$$

$$\{L\} \in \pi_0 \text{Pic } R$$

$$L \underset{R}{\otimes} L \xrightarrow{\text{twist}} L \underset{R}{\otimes} L$$

$$\rightsquigarrow (\pi_0 R)^{\times}[2] \text{ element}$$

Approaches

① Comparison with algebra

② Descent

③ Obstruction theory

Hopkins-Lurie

$$① 0 \rightarrow \underbrace{\text{Pic}(\pi_* R)}_{\psi: N_* \text{ flat } / \pi_* R} \longrightarrow \pi_0 \text{Pic} R$$

Then [Baker-Richter] this is an iso if

a) R is connective

or $\pi_{0, \text{odd}} R = 0$
 $\pi_{0, 2} R$ invertible
 $\pi_{0, R}$ ring

b) R is (weakly) even periodic,
 $\pi_{0, R}$ regular Noetherian

$\pi_{0, R} \cong (\pi_{0, 2} R)^{\otimes k}$

Vistor: much more complicated to have anything like this for Br.
 [Baker-Richter-Szymik,
 Gepner-Lawson]

② Descent

thus $\left[\begin{array}{l} \text{Ando-Blumberg-Gepner-Hopkins-Rezk, } \underline{\text{Antieau-Gepner}} \\ \text{Gepner-Lawson, } \underline{\text{Antieau-Mein-Stojanowicz}} \end{array} \right]$

the functors

$$\begin{array}{ccc}
 \text{Pic} & \text{CAlg}(Sp) & \xrightarrow{\quad} \infty\text{-loop} \\
 \text{Br} & & \text{spaces} \\
 \text{Satisfy} & \nearrow \text{étale descent} & \\
 & \searrow \text{Galois descent} & \\
 \text{Descent: } & \text{e.g. } \text{Pic}(R) = \text{Pic}(S)^{h\mathbb{G}_m} &
 \end{array}$$

$R \rightarrow S$ is étale if

$\pi_0 R \rightarrow \pi_0 S$ is étale, and

$$\pi_* S \xleftarrow{\sim} \pi_* R \otimes_{\pi_0 R} \pi_0 S$$

e.g. KO has no interesting étale extensions

$R \rightarrow S^{2G}$ is G -Galois if

$$\bullet R \xrightarrow{\sim} S^{hG}$$

and

$$\bullet S \underset{R}{\otimes} S \simeq \underset{G}{\prod} S$$

e.g. $KO \rightarrow KU^{\mathbb{Z}/G}$ is G -Galois

$$\pi_* KU = \mathbb{Z}[\frac{p \pm 1}{2}]$$

$$\text{Pic}(\pi_* KU) = \mathbb{Z}/2$$

$$\Rightarrow \text{Pic}(KU) = \mathbb{Z}/2$$

Descent

$$\text{Pic}(KO) \simeq \text{Pic}(KU)$$

needs some
input other
than

$$KO \simeq KU^{hG}$$

Applications

- $R \xrightarrow{\sim} S^{26}$, S even periodic

$\rightsquigarrow \pi_0 \text{pic}(R)$

relative Brauer group

- Étale locally trivial Brauer classes.