Real Bordism, Real orientations, and Lubin–Tate spectra

XiaoLin Danny Shi

University of Chicago

June 2021

What does Real mean?

- C_2: cyclic group of order 2 with generator τ

What does Real mean?

- C_2: cyclic group of order 2 with generator τ
- Real space X: X with a C_2-action

What does Real mean?

- C_2: cyclic group of order 2 with generator τ
- Real space X: X with a C_2-action

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[rr, bend right, "id"'] & X \arrow[r, "\tau"] & X \end{tikzcd}$$

What does Real mean?

- C_2: cyclic group of order 2 with generator τ
- Real space X: X with a C_2-action

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[rr, bend right, "id"'] & X \arrow[r, "\tau"] & X \end{tikzcd}$$

- Real vector bundle E over X:

What does Real mean?

- C_2: cyclic group of order 2 with generator τ
- Real space X: X with a C_2-action

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[rr, bend right, "id"'] & X \arrow[r, "\tau"] & X \end{tikzcd}$$

- Real vector bundle E over X:
  - E: complex vector bundle over X

What does Real mean?

- C_2: cyclic group of order 2 with generator τ
- Real space X: X with a C_2-action

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[rr, bend right, "id"'] & X \arrow[r, "\tau"] & X \end{tikzcd}$$

- Real vector bundle E over X:
  - E: complex vector bundle over X
  - E: Real space

What does Real mean?

- C_2: cyclic group of order 2 with generator τ
- Real space X: X with a C_2-action

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[rr, bend right, "id"'] & X \arrow[r, "\tau"] & X \end{tikzcd}$$

- Real vector bundle E over X:
  - E: complex vector bundle over X
  - E: Real space
  - $p : E \to X$ is C_2-equivariant

What does Real mean?

- C_2: cyclic group of order 2 with generator τ
- Real space X: X with a C_2-action

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[rr, bend right, "id"'] & X \arrow[r, "\tau"] & X \end{tikzcd}$$

- Real vector bundle E over X:
  - E: complex vector bundle over X
  - E: Real space
  - $p : E \to X$ is C_2-equivariant
  - $\tau : E_x \to E_{\tau(x)}$ is anti $\mathbb{C}$-linear

$$\tau(z \cdot v) = \bar{z} \cdot \tau(v)$$

What does Real mean?

- C_2: cyclic group of order 2 with generator τ
- Real space X: X with a C_2-action

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[rr, bend right, "id"'] & X \arrow[r, "\tau"] & X \end{tikzcd}$$

- Real vector bundle E over X:
  - E: complex vector bundle over X
  - E: Real space
  - $p : E \to X$ is C_2-equivariant
  - $\tau : E_x \to E_{\tau(x)}$ is anti $\mathbb{C}$-linear

$$\tau(z \cdot v) = \bar{z} \cdot \tau(v)$$

(This is NOT a C_2-equivariant complex vector bundle!)

What does Real mean?

- C_2: cyclic group of order 2 with generator τ
- Real space X: X with a C_2-action

$$
\begin{tikzcd}
X \arrow[r, "\tau"] \arrow[rr, bend right, "id"'] & X \arrow[r, "\tau"] & X
\end{tikzcd}
$$

- Real vector bundle E over X:
  - E: complex vector bundle over X
  - E: Real space
  - $p : E \to X$ is C_2-equivariant
  - $\tau : E_x \to E_{\tau(x)}$ is anti $\mathbb{C}$-linear

$$\tau(z \cdot v) = \bar{z} \cdot \tau(v)$$

(This is NOT a C_2-equivariant complex vector bundle!)

- $K_{\mathbb{R}}(X)$: Grothendieck's construction

What does Real mean?

- C_2: cyclic group of order 2 with generator τ
- Real space X: X with a C_2-action

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[rr, bend right, "id"'] & X \arrow[r, "\tau"] & X \end{tikzcd}$$

- Real vector bundle E over X:
  - E: complex vector bundle over X
  - E: Real space
  - $p : E \to X$ is C_2-equivariant
  - $\tau : E_x \to E_{\tau(x)}$ is anti $\mathbb{C}$-linear

$$\tau(z \cdot v) = \overline{z} \cdot \tau(v)$$

(This is NOT a C_2-equivariant complex vector bundle!)

- $K_{\mathbb{R}}(X)$: Grothendieck's construction
  $\Longrightarrow$ C_2-equivariant spectrum $K_{\mathbb{R}}$ (Atiyah's Real K-theory)

Equivariant stable homotopy theory

- Non-equivariantly, $\pi_n X = [S^n, X]$

Equivariant stable homotopy theory

- Non-equivariantly, $\pi_n X = [S^n, X]$
- Equivariantly, we have more spheres

Equivariant stable homotopy theory

- Non-equivariantly, $\pi_n X = [S^n, X]$
- Equivariantly, we have more spheres
- V: G-representation $\Longrightarrow S^V$

Equivariant stable homotopy theory

- Non-equivariantly, $\pi_n X = [S^n, X]$
- Equivariantly, we have more spheres
- V: G-representation $\Longrightarrow S^V$
- X: G-spectrum

$$\pi_V^G X = [S^V, X]^G$$

Equivariant stable homotopy theory

- Non-equivariantly, $\pi_n X = [S^n, X]$
- Equivariantly, we have more spheres
- V: G-representation $\Longrightarrow S^V$
- X: G-spectrum

$$\pi_V^G X = [S^V, X]^G$$

- $\pi_\star^G X$: $RO(G)$-graded homotopy groups of X

Atiyah's Real K-theory $K_{\mathbb{R}}$

- $K_{\mathbb{R}}$ combines complex K-theory and real K-theory
  - The underlying spectrum is KU
  - $K_{\mathbb{R}}^{C_2} = K_{\mathbb{R}}^{hC_2} = KO$

Atiyah's Real K-theory $K_{\mathbb{R}}$

- $K_{\mathbb{R}}$ combines complex K-theory and real K-theory
  - The underlying spectrum is KU
  - $K_{\mathbb{R}}^{C_2} = K_{\mathbb{R}}^{hC_2} = KO$
- There are two periodicities:
  - $\pi^{C_2}_{\bigstar+\rho} K_{\mathbb{R}} = \pi^{C_2}_{\bigstar} K_{\mathbb{R}}$ (complex Bott periodicity)
  - $\pi^{C_2}_{\bigstar+8} K_{\mathbb{R}} = \pi^{C_2}_{\bigstar} K_{\mathbb{R}}$ (real Bott periodicity)

Real bordism spectrum $MU_{\mathbb{R}}$

- γ_n: the universal bundle over $BU(n)$

Real bordism spectrum $MU_{\mathbb{R}}$

- γ_n: the universal bundle over $BU(n)$
- $BU(n)^{\gamma_n}$: its Thom space

Real bordism spectrum $MU_{\mathbb{R}}$

- γ_n: the universal bundle over $BU(n)$
- $BU(n)^{\gamma_n}$: its Thom space
- $\Sigma^2 BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$

Real bordism spectrum $MU_{\mathbb{R}}$

- γ_n: the universal bundle over $BU(n)$
- $BU(n)^{\gamma_n}$: its Thom space
- $\Sigma^2 BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$

  $\Longrightarrow$ Thom spectrum MU

Real bordism spectrum $MU_{\mathbb{R}}$

- γ_n: the universal bundle over $BU(n)$
- $BU(n)^{\gamma_n}$: its Thom space
- $\Sigma^2 BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$

  $\Longrightarrow$ Thom spectrum MU
- $BU(n)$: Real space

  γ_n: Real vector bundle

Real bordism spectrum $MU_{\mathbb{R}}$

- γ_n: the universal bundle over $BU(n)$
- $BU(n)^{\gamma_n}$: its Thom space
- $\Sigma^2 BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$

  $\Longrightarrow$ Thom spectrum MU
- $BU(n)$: Real space

  γ_n: Real vector bundle
- $\Sigma^\rho BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$

  ρ: regular representation of C_2

Real bordism spectrum $MU_{\mathbb{R}}$

- γ_n: the universal bundle over $BU(n)$
- $BU(n)^{\gamma_n}$: its Thom space
- $\Sigma^2 BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$

  $\Longrightarrow$ Thom spectrum MU
- $BU(n)$: Real space

  γ_n: Real vector bundle
- $\Sigma^\rho BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$

  ρ: regular representation of C_2

  $\Longrightarrow$ C_2-equivariant Thom spectrum $MU_{\mathbb{R}}$

Real bordism spectrum $MU_{\mathbb{R}}$

- γ_n: the universal bundle over $BU(n)$
- $BU(n)^{\gamma_n}$: its Thom space
- $\Sigma^2 BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$

  $\Longrightarrow$ Thom spectrum MU
- $BU(n)$: Real space

  γ_n: Real vector bundle
- $\Sigma^{\rho} BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$

  ρ: regular representation of C_2

  $\Longrightarrow$ C_2-equivariant Thom spectrum $MU_{\mathbb{R}}$
- The underlying spectrum of $MU_{\mathbb{R}}$ is MU

Real bordism spectrum $MU_{\mathbb{R}}$

- γ_n: the universal bundle over $BU(n)$
- $BU(n)^{\gamma_n}$: its Thom space
- $\Sigma^2 BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$

  $\Longrightarrow$ Thom spectrum MU
- $BU(n)$: Real space

  γ_n: Real vector bundle
- $\Sigma^\rho BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$

  ρ: regular representation of C_2

  $\Longrightarrow$ C_2-equivariant Thom spectrum $MU_{\mathbb{R}}$
- The underlying spectrum of $MU_{\mathbb{R}}$ is MU
- This spectrum is crucial in Hill–Hopkins–Ravenel's solution of the Kervaire invariant one problem (later)

Complex bordism spectrum

- $MU_*(X)$: group of bordism classes of manifolds over X with a complex linear structure on the stable normal bundle

Complex bordism spectrum

- $MU_*(X)$: group of bordism classes of manifolds over X with a complex linear structure on the stable normal bundle
- $\pi_* MU = MU_*(\mathrm{pt}) \cong \mathbb{Z}[x_1, x_2, \ldots]$, where $|x_i| = 2i$ (Milnor, Novikov)

Complex bordism spectrum

- $MU_*(X)$: group of bordism classes of manifolds over X with a complex linear structure on the stable normal bundle
- $\pi_* MU = MU_*(\mathrm{pt}) \cong \mathbb{Z}[x_1, x_2, \ldots]$, where $|x_i| = 2i$ (Milnor, Novikov)
- Associated with MU is a theory of complex orientations

Complex orientation

- E: multiplicative cohomology theory

Complex orientation

- E: multiplicative cohomology theory
- Complex orientation: $x \in \widetilde{E}^2(\mathbf{CP}^\infty)$, restricts to the unit in

$$\widetilde{E}^2(\mathbf{CP}^1) = \widetilde{E}^2(S^2) \cong E^0(\mathrm{pt})$$

Complex orientation

- E: multiplicative cohomology theory
- Complex orientation: $x \in \widetilde{E}^2(\mathbf{CP}^\infty)$, restricts to the unit in

$$\widetilde{E}^2(\mathbf{CP}^1) = \widetilde{E}^2(S^2) \cong E^0(\mathrm{pt})$$

- $E^*(\mathbf{CP}^\infty) \cong E^*[\![x]\!]$
  $E^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \cong E^*[\![x, y]\!]$

Complex orientation

- E: multiplicative cohomology theory
- Complex orientation: $x \in \widetilde{E}^2(\mathbf{CP}^\infty)$, restricts to the unit in
$$\widetilde{E}^2(\mathbf{CP}^1) = \widetilde{E}^2(S^2) \cong E^0(\mathrm{pt})$$
- $E^*(\mathbf{CP}^\infty) \cong E^*[\![x]\!]$
  $E^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \cong E^*[\![x, y]\!]$
- $\mathbf{CP}^\infty \times \mathbf{CP}^\infty \longrightarrow \mathbf{CP}^\infty$
  (classifying tensor product of tautological line bundles)

Complex orientation

- E: multiplicative cohomology theory
- Complex orientation: $x \in \widetilde{E}^2(\mathbf{CP}^\infty)$, restricts to the unit in
$$\widetilde{E}^2(\mathbf{CP}^1) = \widetilde{E}^2(S^2) \cong E^0(\mathrm{pt})$$
- $E^*(\mathbf{CP}^\infty) \cong E^*[\![x]\!]$
$E^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \cong E^*[\![x, y]\!]$
- $\mathbf{CP}^\infty \times \mathbf{CP}^\infty \longrightarrow \mathbf{CP}^\infty$
(classifying tensor product of tautological line bundles)
- $E^*(\mathbf{CP}^\infty) \longrightarrow E^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty)$
$x \longmapsto F(x, y) \Longrightarrow$ formal group law / $\pi_* E$

Complex orientation

- Complex orientation $\Longleftrightarrow (MU \longrightarrow E)$

Complex orientation

- Complex orientation $\Longleftrightarrow (MU \longrightarrow E)$
- Example: $(MU \longrightarrow H\mathbb{Z}) \Longrightarrow \mathbb{G}_a(x, y) = x + y$

Complex orientation

- Complex orientation $\Longleftrightarrow (MU \longrightarrow E)$
- Example: $(MU \longrightarrow H\mathbb{Z}) \Longrightarrow \mathbb{G}_a(x,y) = x + y$
- Example: $(MU \longrightarrow KU) \Longrightarrow \mathbb{G}_m(x,y) = x + y - \beta xy$

Complex orientation

- Complex orientation $\Longleftrightarrow (MU \longrightarrow E)$
- Example: $(MU \longrightarrow H\mathbb{Z}) \Longrightarrow \mathbb{G}_a(x, y) = x + y$
- Example: $(MU \longrightarrow KU) \Longrightarrow \mathbb{G}_m(x, y) = x + y - \beta xy$

Theorem (Quillen)

MU itself is complex oriented and it carries the universal formal group law.

Real orientation

- Associated to $MU_{\mathbb{R}}$ is a theory of Real orientations

Real orientation

- Associated to $MU_{\mathbb{R}}$ is a theory of Real orientations
- E: C_2-equivariant homotopy commutative ring spectrum

Real orientation

- Associated to $MU_{\mathbb{R}}$ is a theory of Real orientations
- E: C_2-equivariant homotopy commutative ring spectrum
- Real orientation: $\bar{x} \in \widetilde{E}^{\rho}_{C_2}(\mathbf{CP}^{\infty})$, restricts to the unit in

$$\widetilde{E}^{\rho}_{C_2}(\mathbf{CP}^1) = \widetilde{E}^{\rho}_{C_2}(S^{\rho}) \cong E^0_{C_2}(\mathrm{pt})$$

Real orientation

- Associated to $MU_{\mathbb{R}}$ is a theory of Real orientations
- E: C_2-equivariant homotopy commutative ring spectrum
- Real orientation: $\bar{x} \in \widetilde{E}^{\rho}_{C_2}(\mathbf{CP}^{\infty})$, restricts to the unit in

$$\widetilde{E}^{\rho}_{C_2}(\mathbf{CP}^1) = \widetilde{E}^{\rho}_{C_2}(S^{\rho}) \cong E^0_{C_2}(\mathrm{pt})$$

- $E^{\bigstar}_{C_2}(\mathbf{CP}^{\infty}) \cong E^{\bigstar}_{C_2}[\![\bar{x}]\!]$
  $E^{\bigstar}_{C_2}(\mathbf{CP}^{\infty} \times \mathbf{CP}^{\infty}) \cong E^{\bigstar}_{C_2}[\![\bar{x}, \bar{y}]\!]$ (Araki)

Real orientation

- Associated to $MU_{\mathbb{R}}$ is a theory of Real orientations
- E: C_2-equivariant homotopy commutative ring spectrum
- Real orientation: $\bar{x} \in \widetilde{E}^{\rho}_{C_2}(\mathbf{CP}^{\infty})$, restricts to the unit in

$$\widetilde{E}^{\rho}_{C_2}(\mathbf{CP}^1) = \widetilde{E}^{\rho}_{C_2}(S^{\rho}) \cong E^0_{C_2}(\mathrm{pt})$$

- $E^{\bigstar}_{C_2}(\mathbf{CP}^{\infty}) \cong E^{\bigstar}_{C_2}[\![\bar{x}]\!]$
  $E^{\bigstar}_{C_2}(\mathbf{CP}^{\infty} \times \mathbf{CP}^{\infty}) \cong E^{\bigstar}_{C_2}[\![\bar{x}, \bar{y}]\!]$ (Araki)
- $\mathbf{CP}^{\infty} \times \mathbf{CP}^{\infty} \longrightarrow \mathbf{CP}^{\infty} \Longrightarrow$ formal group law $/\, \pi^{C_2}_{\bigstar} E$

Real orientation

- Associated to $MU_{\mathbb{R}}$ is a theory of Real orientations
- E: C_2-equivariant homotopy commutative ring spectrum
- Real orientation: $\bar{x} \in \widetilde{E}^{\rho}_{C_2}(\mathbf{CP}^{\infty})$, restricts to the unit in

$$\widetilde{E}^{\rho}_{C_2}(\mathbf{CP}^1) = \widetilde{E}^{\rho}_{C_2}(S^{\rho}) \cong E^0_{C_2}(\mathrm{pt})$$

- $E^{\bigstar}_{C_2}(\mathbf{CP}^{\infty}) \cong E^{\bigstar}_{C_2}[\![\bar{x}]\!]$
  $E^{\bigstar}_{C_2}(\mathbf{CP}^{\infty} \times \mathbf{CP}^{\infty}) \cong E^{\bigstar}_{C_2}[\![\bar{x}, \bar{y}]\!]$ (Araki)
- $\mathbf{CP}^{\infty} \times \mathbf{CP}^{\infty} \longrightarrow \mathbf{CP}^{\infty} \Longrightarrow$ formal group law $/\, \pi^{C_2}_{\bigstar} E$
- Real orientation $\Longleftrightarrow$ $(MU_{\mathbb{R}} \longrightarrow E)$ (Hu–Kriz)

Real orientation

- Associated to $MU_{\mathbb{R}}$ is a theory of Real orientations
- E: C_2-equivariant homotopy commutative ring spectrum
- Real orientation: $\bar{x} \in \widetilde{E}^{\rho}_{C_2}(\mathbf{CP}^\infty)$, restricts to the unit in

$$\widetilde{E}^{\rho}_{C_2}(\mathbf{CP}^1) = \widetilde{E}^{\rho}_{C_2}(S^\rho) \cong E^0_{C_2}(\mathrm{pt})$$

- $E^{\bigstar}_{C_2}(\mathbf{CP}^\infty) \cong E^{\bigstar}_{C_2}[\![\bar{x}]\!]$
  $E^{\bigstar}_{C_2}(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \cong E^{\bigstar}_{C_2}[\![\bar{x}, \bar{y}]\!]$ (Araki)
- $\mathbf{CP}^\infty \times \mathbf{CP}^\infty \longrightarrow \mathbf{CP}^\infty \Longrightarrow$ formal group law $/\, \pi^{C_2}_{\bigstar} E$
- Real orientation $\Longleftrightarrow$ $(MU_{\mathbb{R}} \longrightarrow E)$ (Hu–Kriz)
- Example: $MU_{\mathbb{R}} \longrightarrow K_{\mathbb{R}}$

Lubin–Tate spectra

- KU belongs to a more general class of spectra

Lubin–Tate spectra

- KU belongs to a more general class of spectra
- k: perfect field, characteristic p

Lubin–Tate spectra

- KU belongs to a more general class of spectra
- k: perfect field, characteristic p
- Γ_n: formal group law of height n over k

Lubin–Tate spectra

- KU belongs to a more general class of spectra
- k: perfect field, characteristic p
- Γ_n: formal group law of height n over k
- Lubin–Tate: Γ_n admits an universal deformation
  - Characterized by a map $MU_* \longrightarrow E_{n*}$
  - $E_{n*} \cong W(k)[\![u_1, u_2, \ldots, u_{n-1}]\!][u^{\pm}]$ is the universal deformation ring

Lubin–Tate spectra

- KU belongs to a more general class of spectra
- k: perfect field, characteristic p
- Γ_n: formal group law of height n over k
- Lubin–Tate: Γ_n admits an universal deformation
  - Characterized by a map $MU_* \longrightarrow E_{n*}$
  - $E_{n*} \cong W(k)[\![u_1, u_2, \dots, u_{n-1}]\!][u^{\pm}]$ is the universal deformation ring
- Landweber exact functor theorem
  $\Longrightarrow E_n$: Complex oriented cohomology theory

Lubin–Tate spectra

- KU belongs to a more general class of spectra
- k: perfect field, characteristic p
- Γ_n: formal group law of height n over k
- Lubin–Tate: Γ_n admits an universal deformation
  - Characterized by a map $MU_* \longrightarrow E_{n*}$
  - $E_{n*} \cong W(k)[\![u_1, u_2, \ldots, u_{n-1}]\!][u^{\pm}]$ is the universal deformation ring
- Landweber exact functor theorem
  $\Longrightarrow E_n$: Complex oriented cohomology theory
- $k = \mathbb{F}_2$, $\Gamma_1(x, y) = x + y - \beta xy \Longrightarrow E_1 = KU_2^{\wedge}$

Lubin–Tate spectra

- KU belongs to a more general class of spectra
- k: perfect field, characteristic p
- Γ_n: formal group law of height n over k
- Lubin–Tate: Γ_n admits an universal deformation
  - Characterized by a map $MU_* \longrightarrow E_{n*}$
  - $E_{n*} \cong W(k)[\![u_1, u_2, \ldots, u_{n-1}]\!][u^{\pm}]$ is the universal deformation ring
- Landweber exact functor theorem
  $\Longrightarrow E_n$: Complex oriented cohomology theory
- $k = \mathbb{F}_2$, $\Gamma_1(x, y) = x + y - \beta xy \Longrightarrow E_1 = KU_2^{\wedge}$
- Formal group laws associated with elliptic curves
  $\Longrightarrow$ Elliptic cohomology theories
  (topological modular forms, string orientations, Witten genus)

Application: stable homotopy groups of spheres

Theorem (Frudenthal 1937)

$\pi_{n+k}(S^n)$ *stabilizes for* $n > k+1 \Longrightarrow \pi_k^{st}(S^0)$

Application: stable homotopy groups of spheres

Theorem (Frudenthal 1937)

$\pi_{n+k}(S^n)$ *stabilizes for* $n > k + 1 \Longrightarrow \pi_k^{st}(S^0)$

Definition

$\pi_k^{st}(S^0) := \varinjlim \pi_{n+k}(S^n)$

Application: stable homotopy groups of spheres

Theorem (Pontryagin 1930s)

$$\Omega_k^{fr} \cong \pi_k^{st}(S^0)$$

Ω_k^{fr}: cobordism classes of stably framed k-manifolds

Application: stable homotopy groups of spheres

Theorem (Pontryagin 1930s)

$$\Omega_k^{fr} \cong \pi_k^{st}(S^0)$$

Ω_k^{fr}*: cobordism classes of stably framed k-manifolds*

- This establishes a deep relationship between homotopy theory and geometry

Application: stable homotopy groups of spheres

Theorem (Pontryagin 1930s)

$$\Omega_k^{fr} \cong \pi_k^{st}(S^0)$$

Ω_k^{fr}: *cobordism classes of stably framed k-manifolds*

- This establishes a deep relationship between homotopy theory and geometry
- Describing $\pi_*^{st} S^0$ has been fundamental to algebraic topology for the past 80 years

Application: stable homotopy groups of spheres

Theorem (Pontryagin 1930s)

$$\Omega_k^{fr} \cong \pi_k^{st}(S^0)$$

Ω_k^{fr}: *cobordism classes of stably framed k-manifolds*

- This establishes a deep relationship between homotopy theory and geometry
- Describing $\pi_*^{st} S^0$ has been fundamental to algebraic topology for the past 80 years
- Lubin–Tate spectra can isolate certain “sectors” of the computation + give connections to others areas (modular forms, geometric topology)

Lubin–Tate spectra

$$\begin{tikzcd} E_n \arrow[loop above, "\mathbb{G}(k,\Gamma_n)"] \arrow[d, dashed] \\ E_{n*} \arrow[loop below, "\mathbb{G}(k,\Gamma_n) = \mathrm{Aut}(\Gamma_n) \rtimes \mathrm{Gal}(k/\mathbb{F}_p)"] \end{tikzcd}$$

Lubin–Tate spectra

$$\begin{tikzcd}
E_n \arrow[loop above, "\mathbb{G}(k,\Gamma_n)"] \arrow[d, dashed] \\
E_{n*} \arrow[loop below, "\mathbb{G}(k,\Gamma_n) = \mathrm{Aut}(\Gamma_n) \rtimes \mathrm{Gal}(k/\mathbb{F}_p)"]
\end{tikzcd}$$

Theorem (Goerss–Hopkins–Miller)

The action of $\mathbb{G}(k, \Gamma_n)$ on E_{n*} can be lifted uniquely to an action of $\mathbb{G}(k, \Gamma_n)$ on E_n by commutative ($\mathbb{E}_\infty$) ring maps.

Lubin–Tate spectra

$$
\begin{tikzcd}
E_n \arrow[loop above, "\mathbb{G}(k,\Gamma_n)"] \arrow[d, dashed] \\
E_{n*} \arrow[loop below, "\mathbb{G}(k,\Gamma_n)=\mathrm{Aut}(\Gamma_n)\rtimes\mathrm{Gal}(k/\mathbb{F}_p)"]
\end{tikzcd}
$$

Theorem (Goerss–Hopkins–Miller)

The action of $\mathbb{G}(k,\Gamma_n)$ on E_{n*} can be lifted uniquely to an action of $\mathbb{G}(k,\Gamma_n)$ on E_n by commutative ($\mathbb{E}_\infty$) ring maps.

- Can view E_n as a $\mathbb{G}(k,\Gamma_n)$-equivariant commutative ring spectrum

C_2-action

- From now on: $p = 2$

C_2-action

- From now on: $p = 2$
- $C_2 \subset \mathbb{G}(k, \Gamma_n)$: acts on E_{n*} by $[-1]_{\Gamma_n}$.

C_2-action

- From now on: $p = 2$
- $C_2 \subset \mathbb{G}(k, \Gamma_n)$: acts on E_{n*} by $[-1]_{\Gamma_n}$.
- $$\begin{tikzcd} E_n \arrow[loop above, "C_2"] \end{tikzcd}$$

Question

$$\begin{tikzcd} MU_* \arrow[r] \arrow[loop below, "C_2"] & E_{n*} \arrow[loop below, "C_2"] \end{tikzcd}$$

Question

$$
\begin{tikzcd}
\text{Geometry} & \text{Algebra} \\
MU_{\mathbb{R}} \arrow[loop above, "C_2"] \arrow[d, dashed] & E_n \arrow[loop above, "C_2"] \arrow[d, dashed] \\
MU_* \arrow[loop below, "C_2"'] \arrow[r] & E_{n*} \arrow[loop below, "C_2"']
\end{tikzcd}
$$

Question

Geometry Algebra

$$\begin{tikzcd} MU_{\mathbb{R}} \arrow[loop above, "C_2"] \arrow[r, dashed, red, "?"] \arrow[d, dashed] & E_n \arrow[loop above, "C_2"] \arrow[d, dashed] \\ MU_* \arrow[loop below, "C_2"] \arrow[r] & E_{n*} \arrow[loop below, "C_2"] \end{tikzcd}$$

Can we lift it?

Real Orientation

Theorem (Hahn–S.)

The Lubin–Tate spectrum E_n is Real oriented: it receives a C_2-equivariant map

$$MU_{\mathbb{R}} \longrightarrow E_n$$

from the Real bordism spectrum $MU_{\mathbb{R}}$.

Real Orientation

Theorem (Hahn–S.)

The Lubin–Tate spectrum E_n is Real oriented: it receives a C_2-equivariant map

$$MU_{\mathbb{R}} \longrightarrow E_n$$

from the Real bordism spectrum $MU_{\mathbb{R}}$.

This opens the door to a series of computations in stable homotopy theory

Chromatic homotopy theory

Theorem (Hopkins–Ravenel, Chromatic Convergence)

$$S^0_{(p)} \xrightarrow{\simeq} \cdots \longrightarrow L_{E_n}S^0 \longrightarrow L_{E_{n-1}}S^0 \longrightarrow \cdots \longrightarrow L_{E_0}S^0.$$

Chromatic homotopy theory

Theorem (Hopkins–Ravenel, Chromatic Convergence)

$$S^0_{(p)} \xrightarrow{\simeq} \cdots \longrightarrow L_{E_n}S^0 \longrightarrow L_{E_{n-1}}S^0 \longrightarrow \cdots \longrightarrow L_{E_0}S^0.$$

Chromatic fracture square:

$$\begin{tikzcd}
L_{E_n}S^0 \arrow[r] \arrow[d] & L_{K(n)}S^0 \arrow[d] \\
L_{E_{n-1}}S^0 \arrow[r] & L_{E_{n-1}}L_{K(n)}S^0
\end{tikzcd}$$

Chromatic homotopy theory

Theorem (Hopkins–Ravenel, Chromatic Convergence)

$$S^0_{(p)} \xrightarrow{\simeq} \cdots \longrightarrow L_{E_n}S^0 \longrightarrow L_{E_{n-1}}S^0 \longrightarrow \cdots \longrightarrow L_{E_0}S^0.$$

Chromatic fracture square:

$$\begin{tikzcd} L_{E_n}S^0 \arrow[r] \arrow[d] & L_{K(n)}S^0 \arrow[d] \\ L_{E_{n-1}}S^0 \arrow[r] & L_{E_{n-1}}L_{K(n)}S^0 \end{tikzcd}$$

Slogan: in order to study $S^0_{(p)}$, we just need to study each of the $K(n)$-local spheres and how they "glue" together

Chromatic homotopy theory

Theorem (Hopkins–Devinatz)

$$L_{K(n)}S^0 \xrightarrow{\simeq} E_n^{h\mathbb{G}_n}$$

- E_n^{hG} (G a finite subgroup of $\mathbb{G}_n$): central objects to study in chromatic homotopy theory

Chromatic homotopy theory

Theorem (Hopkins–Devinatz)

$$L_{K(n)}S^0 \xrightarrow{\simeq} E_n^{h\mathbb{G}_n}$$

- E_n^{hG} (G a finite subgroup of $\mathbb{G}_n$): central objects to study in chromatic homotopy theory
- In general, E_n^{hG} capture large scale periodicity phenomena in stable homotopy theory (in particular $\pi_* S^0$)

Chromatic homotopy theory

Theorem (Hopkins–Devinatz)

$$L_{K(n)}S^0 \xrightarrow{\simeq} E_n^{h\mathbb{G}_n}$$

- E_n^{hG} (G a finite subgroup of $\mathbb{G}_n$): central objects to study in chromatic homotopy theory
- In general, E_n^{hG} capture large scale periodicity phenomena in stable homotopy theory (in particular $\pi_* S^0$)
- Modern detection theorems: study elements in $\pi_* S^0$ by analyzing $\pi_* S^0 \longrightarrow \pi_* E_n^{hG}$ (Hill–Hopkins–Ravenel's solution to the Kervaire invariant)

Computing E_n^{hG}: height 1

- Homotopy fixed point spectral sequence:
  $H^*(G; \pi_* E_n) \Longrightarrow \pi_* E_n^{hG}$

Computing E_n^{hG}: height 1

- Homotopy fixed point spectral sequence: $H^*(G; \pi_* E_n) \Longrightarrow \pi_* E_n^{hG}$
- Height 1: $E_1^{hC_2} = KO_2^\wedge$
  - Image of $J : \pi_* O \to \pi_* S^0$
  - Captures everything above a line of slope $\frac{1}{5}$ in the Adams–Novikov spectral sequence of S^0 (Mahowald)

Height 2: tmf

- Height 2: tmf, tmf with level structures

Height 2: tmf

- Height 2: tmf, tmf with level structures
- tmf: topological refinement of the classical ring of integral modular forms

Height 2: tmf

- Height 2: tmf, tmf with level structures
- tmf: topological refinement of the classical ring of integral modular forms
- Very active area of research for the past 30 years (String orientation of tmf refines the Witten genus)

Height 2: tmf

- Height 2: tmf, tmf with level structures
- tmf: topological refinement of the classical ring of integral modular forms
- Very active area of research for the past 30 years (String orientation of tmf refines the Witten genus)
- The Hurewciz map $\pi_* S^0 \longrightarrow \pi_* \mathrm{tmf}$ detects an astounding number of elements (nearly all of $\pi_* S^0$ for $* < 60$)

Height 2: tmf

- Height 2: tmf, tmf with level structures
- tmf: topological refinement of the classical ring of integral modular forms
- Very active area of research for the past 30 years (String orientation of tmf refines the Witten genus)
- The Hurewciz map $\pi_* S^0 \longrightarrow \pi_*\mathrm{tmf}$ detects an astounding number of elements (nearly all of $\pi_* S^0$ for $* < 60$)
- They resolve the $K(2)$-local sphere

Height 2: tmf

- Height 2: tmf, tmf with level structures
- tmf: topological refinement of the classical ring of integral modular forms
- Very active area of research for the past 30 years (String orientation of tmf refines the Witten genus)
- The Hurewciz map $\pi_* S^0 \longrightarrow \pi_*\mathrm{tmf}$ detects an astounding number of elements (nearly all of $\pi_* S^0$ for $* < 60$)
- They resolve the $K(2)$-local sphere
- These computations rely heavily on the geometry of elliptic curves
  - Choose a specific super-singular elliptic curve
  - Have a good understanding of how G is acting on $\pi_* E_2$

Height > 2

- $n > 2$: E_n^{hG} sees more, but they are extremely difficult to compute

Height > 2

- $n > 2$: E_n^{hG} sees more, but they are extremely difficult to compute
- Group actions on E_n: constructed purely from obstruction theory

Height > 2

- $n > 2$: E_n^{hG} sees more, but they are extremely difficult to compute
- Group actions on E_n: constructed purely from obstruction theory
- Elliptic curves are not available at higher heights

Height > 2

- $n > 2$: E_n^{hG} sees more, but they are extremely difficult to compute
- Group actions on E_n: constructed purely from obstruction theory
- Elliptic curves are not available at higher heights
- There have been attempts to understanding this using TAF (Behrens–Lawson)

Height > 2

- $n > 2$: E_n^{hG} sees more, but they are extremely difficult to compute
- Group actions on E_n: constructed purely from obstruction theory
- Elliptic curves are not available at higher heights
- There have been attempts to understanding this using TAF (Behrens–Lawson)
- However, the group action on $MU_{\mathbb{R}}$ does come from geometry

Height > 2

- $n > 2$: E_n^{hG} sees more, but they are extremely difficult to compute
- Group actions on E_n: constructed purely from obstruction theory
- Elliptic curves are not available at higher heights
- There have been attempts to understanding this using TAF (Behrens–Lawson)
- However, the group action on $MU_{\mathbb{R}}$ does come from geometry
- The Real orientation establishes a connection between these two actions!

Computation of $E_n^{hC_2}$ at all heights

$$MU_{\mathbb{R}} \longrightarrow E_n$$

Computation of $E_n^{hC_2}$ at all heights

$$\mathrm{HFPSS}(MU_{\mathbb{R}}) \longrightarrow \mathrm{HFPSS}(E_n)$$

Computation of $E_n^{hC_2}$ at all heights

$$\mathrm{HFPSS}(MU_{\mathbb{R}}) \longrightarrow \mathrm{HFPSS}(E_n)$$

- $d_3(u_{2\sigma}) = \bar{x}_1 a_\sigma^3$

Computation of $E_n^{hC_2}$ at all heights

$$\mathrm{HFPSS}(MU_{\mathbb{R}}) \longrightarrow \mathrm{HFPSS}(E_n)$$

- $d_3(u_{2\sigma}) = \bar{x}_1 a_\sigma^3$
- $d_7(u_{4\sigma}) = \bar{x}_3 a_\sigma^7$

Computation of $E_n^{hC_2}$ at all heights

$$\mathrm{HFPSS}(MU_{\mathbb{R}}) \longrightarrow \mathrm{HFPSS}(E_n)$$

- $d_3(u_{2\sigma}) = \bar{x}_1 a_\sigma^3$
- $d_7(u_{4\sigma}) = \bar{x}_3 a_\sigma^7$
- $d_{15}(u_{8\sigma}) = \bar{x}_7 a_\sigma^{15}$

Computation of $E_n^{hC_2}$ at all heights

$$\mathrm{HFPSS}(MU_{\mathbb{R}}) \longrightarrow \mathrm{HFPSS}(E_n)$$

- $d_3(u_{2\sigma}) = \bar{x}_1 a_\sigma^3$
- $d_7(u_{4\sigma}) = \bar{x}_3 a_\sigma^7$
- $d_{15}(u_{8\sigma}) = \bar{x}_7 a_\sigma^{15}$
- $d_{2^{k+1}-1}(u_{2\sigma}^{2^{k-1}}) = \bar{x}_{2^k-1} a_\sigma^{2^{k+1}-1}$

Computation of $E_n^{hC_2}$ at all heights

$$\mathrm{HFPSS}(MU_{\mathbb{R}}) \longrightarrow \mathrm{HFPSS}(E_n)$$

- $d_3(u_{2\sigma}) = \bar{x}_1 a_\sigma^3$
- $d_7(u_{4\sigma}) = \bar{x}_3 a_\sigma^7$
- $d_{15}(u_{8\sigma}) = \bar{x}_7 a_\sigma^{15}$
- $d_{2^{k+1}-1}(u_{2\sigma}^{2^{k-1}}) = \bar{x}_{2^k-1} a_\sigma^{2^{k+1}-1}$
- These differentials induce all the differentials in HFPSS(E_n)

Computation of $E_n^{hC_2}$ at all heights

Theorem (Hahn–S.)

The E_2-page of the $RO(C_2)$-graded homotopy fixed point spectral sequence of E_n is

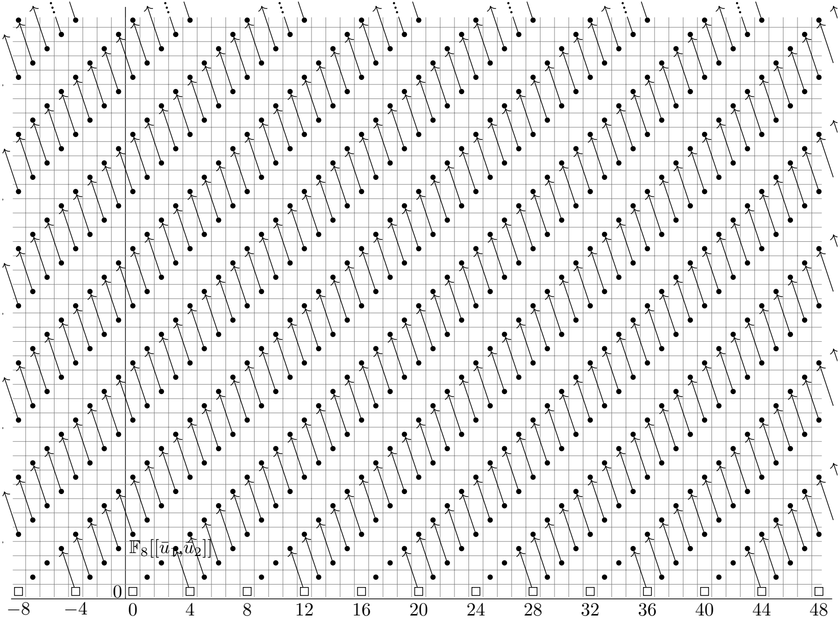
$$E_2^{s,t}(E_n^{hC_2}) = W(\mathbb{F}_{2^n})[\![\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-1}]\!][\bar{u}^{\pm}] \otimes \mathbb{Z}[u_{2\sigma}^{\pm}, a_\sigma]/(2a_\sigma).$$

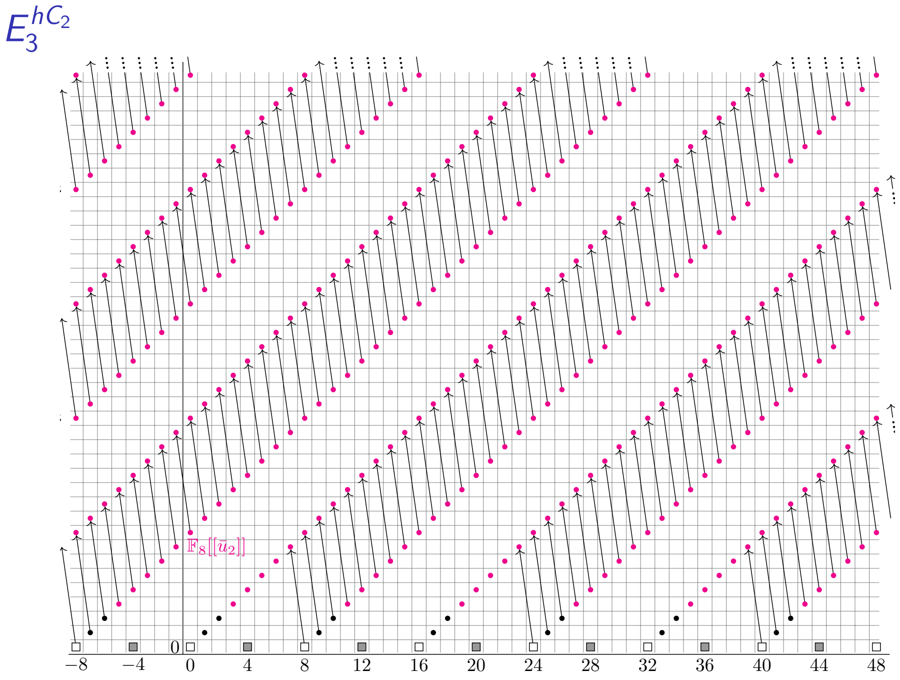
The classes $\bar{u}_1$, …, $\bar{u}_{n-1}$, $\bar{u}^{\pm}$, and a_σ are permanent cycles. All the differentials in the spectral sequence are determined by the differentials

$$\begin{aligned} d_{2^{k+1}-1}(u_{2\sigma}^{2^{k-1}}) &= \bar{u}_k \bar{u}^{2^k-1} a_\sigma^{2^{k+1}-1}, \ \ 1 \leq k \leq n-1, \\ d_{2^{n+1}-1}(u_{2\sigma}^{2^{n-1}}) &= \bar{u}^{2^n-1} a_\sigma^{2^{n+1}-1}, \ \ k = n, \end{aligned}$$

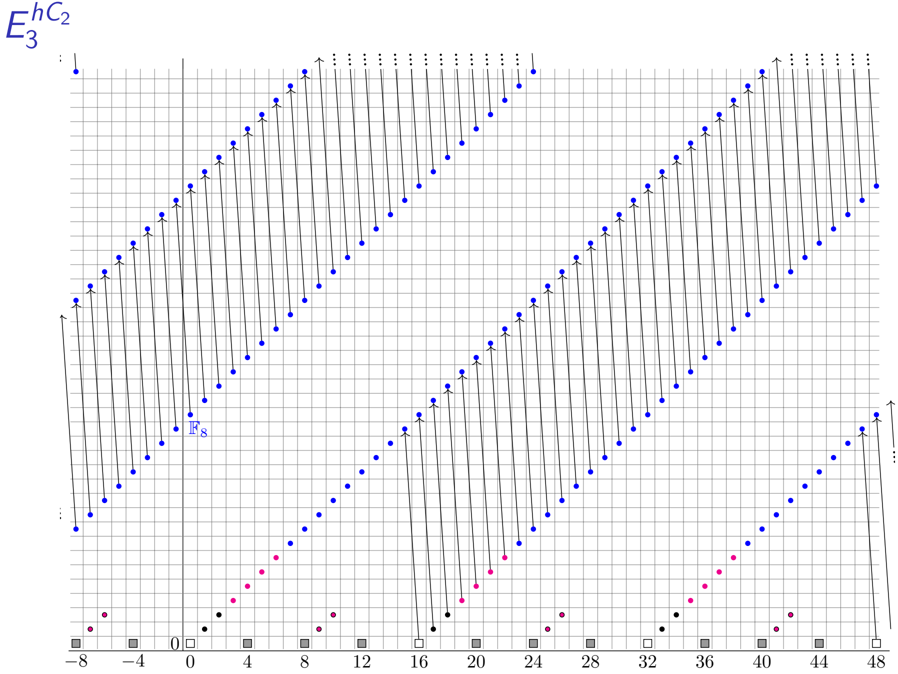
and multiplicative structures.

$E_3^{hC_2}$

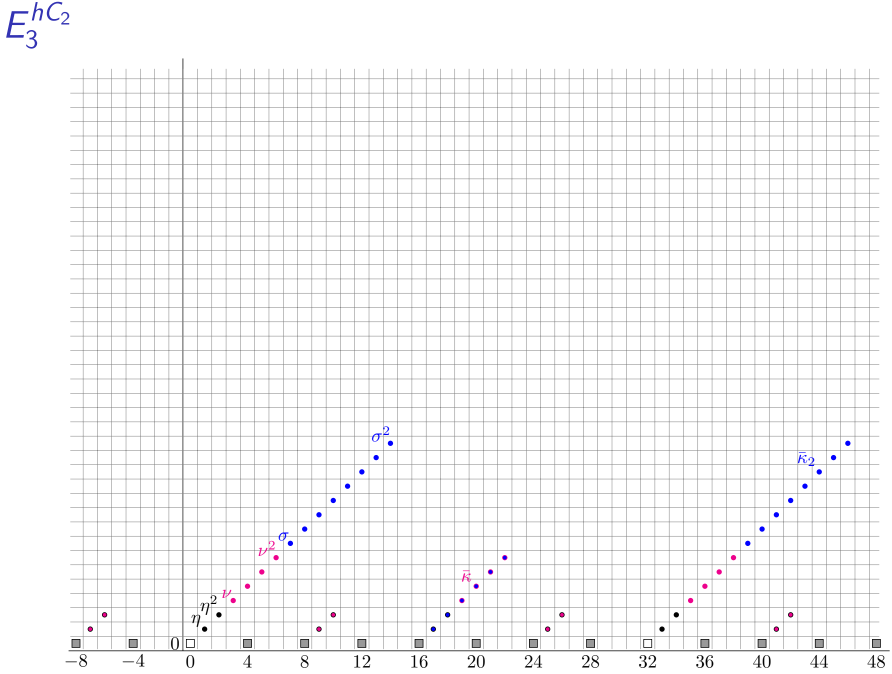




$E_3^{hC_2}$
$\mathbb{F}_8[[\bar{u}_2]]$
0
−8
−4
0
4
8
12
16
20
24
28
32
36
40
44
48



$E_3^{hC_2}$
$\mathbb{F}_8$
0
-8
-4
0
4
8
12
16
20
24
28
32
36
40
44
48


$E_3^{hC_2}$
σ^2
$\bar{\kappa}_2$
σ
ν^2
$\bar{\kappa}$
ν
η^2
η
0
−8 −4 0 4 8 12 16 20 24 28 32 36 40 44 48

Theorem (Hahn–S.)

- *$\pi_* E_n^{hC_2}$ is 2^{n+2}-periodic for all n.*

Theorem (Hahn–S.)

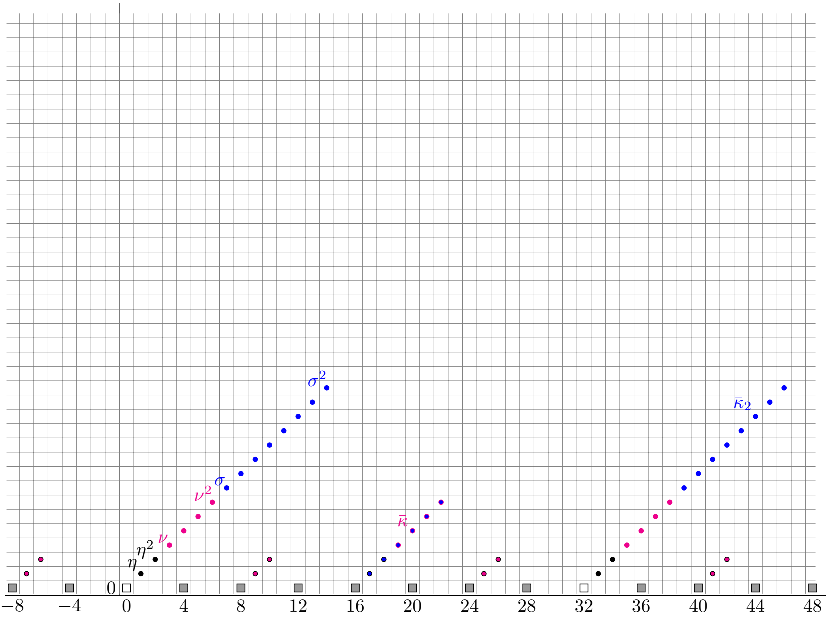
- *$\pi_* E_n^{hC_2}$ is 2^{n+2}-periodic for all n.*
- *E_n is C_2-equivariantly even:*
  - *$\underline{\pi}_{k\rho-1} E_n = 0$ for all $k \in \mathbb{Z}$;*
  - *$\underline{\pi}_{k\rho} E_n$ is a constant Mackey functor for all $k \in \mathbb{Z}$*

Theorem (Hahn–S.)

- *$\pi_* E_n^{hC_2}$ is 2^{n+2}-periodic for all n.*
- *E_n is C_2-equivariantly even:*
  - *$\underline{\pi}_{k\rho-1} E_n = 0$ for all $k \in \mathbb{Z}$;*
  - *$\underline{\pi}_{k\rho} E_n$ is a constant Mackey functor for all $k \in \mathbb{Z}$*
- *E_n is Real Landweber exact:*

$$MU_{\mathbb{R}\bigstar}(X) \otimes_{MU_*} E_{n*} \longrightarrow E_{n\bigstar}(X)$$

is an isomorphism for every C_2-spectrum X.



σ^2
$\bar{\kappa}_2$
σ
ν^2
$\bar{\kappa}$
ν
η^2
η
0
−8
−4
0
4
8
12
16
20
24
28
32
36
40
44
48

Hurewicz image

Theorem (Li–S.–Wang–Xu)

The C_2-fixed points of $MU_{\mathbb{R}}$ detects the Hopf-, Kervaire-, and $\bar{\kappa}$-family.

Theorem (Li–S.–Wang–Xu, Hahn–S.)

The C_2-fixed points of E_n detects the first n elements of the Hopf- and Kervaire-family, and the first $(n-1)$ elements of the $\bar{\kappa}$-family.

The Norm

- What about E_n^{hG} for higher groups?

The Norm

- What about E_n^{hG} for higher groups?
- $H \subset G$ a subgoup

The Norm

- What about E_n^{hG} for higher groups?
- $H \subset G$ a subgoup
- Norm functor $N_H^G : \mathcal{S}^H \longrightarrow \mathcal{S}^G$ (Hill–Hopkins–Ravenel)

The Norm

- What about E_n^{hG} for higher groups?
- $H \subset G$ a subgoup
- Norm functor $N_H^G : \mathcal{S}^H \longrightarrow \mathcal{S}^G$ (Hill–Hopkins–Ravenel)
- G: a group that contains C_2

The Norm

- What about E_n^{hG} for higher groups?
- $H \subset G$ a subgoup
- Norm functor $N_H^G : \mathcal{S}^H \longrightarrow \mathcal{S}^G$ (Hill–Hopkins–Ravenel)
- G: a group that contains C_2
- $MU^{((G))} := N_{C_2}^G MU_{\mathbb{R}}$

G-orientation

Theorem (Hahn–S.)

Let $G \subset \mathbb{G}(k, \Gamma_n)$ be a finite subgroup containing the central subgroup C_2. There is a G-equivariant map

$$MU^{((G))} \longrightarrow E_n.$$

G-orientation

Theorem (Hahn–S.)

Let $G \subset \mathbb{G}(k, \Gamma_n)$ be a finite subgroup containing the central subgroup C_2. There is a G-equivariant map

$$MU^{(\!(G)\!)} \longrightarrow E_n.$$

Motivation: $MU^{(\!(G)\!)}$ is crucial in Hill–Hopkins–Ravenel's solution of the Kervaire invariant one problem

The Kervaire invariant

- M: framed $(4k+2)$-dimensional manifold

The Kervaire invariant

- M: framed $(4k+2)$-dimensional manifold
- Kervaire used the framing to construct a quadratic form

$$\phi : H^{2k+1}(M;\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

$$\phi(x+y) = \phi(x) + \phi(y) + \langle x, y \rangle$$

The Kervaire invariant

- M: framed $(4k+2)$-dimensional manifold
- Kervaire used the framing to construct a quadratic form

$$\phi : H^{2k+1}(M; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

$$\phi(x+y) = \phi(x) + \phi(y) + \langle x, y \rangle$$

- The Kervaire invariant of M is defined as $\Phi(M) := Arf(\phi)$

The Kervaire invariant

- M: framed $(4k+2)$-dimensional manifold
- Kervaire used the framing to construct a quadratic form

$$\phi : H^{2k+1}(M;\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

$$\phi(x+y) = \phi(x) + \phi(y) + \langle x, y\rangle$$

- The Kervaire invariant of M is defined as $\Phi(M) := Arf(\phi)$
- A fundamental invariant in differential and algebraic topology

Smooth structures on S^n

Definition

A homotopy n-sphere: closed manifold $\Sigma^n \simeq S^n$

Smooth structures on S^n

Definition

A homotopy n-sphere: closed manifold $\Sigma^n \simeq S^n$

Question

Is Σ^n *homeomorphic to* S^n*?*

Smooth structures on S^n

Definition

A homotopy n-sphere: closed manifold $\Sigma^n \simeq S^n$

Question

Is Σ^n *homeomorphic to* S^n*?*

- Yes. (Generalized) Poincaré conjecture

Smooth structures on S^n

Definition

A homotopy n-sphere: closed manifold $\Sigma^n \simeq S^n$

Question

Is Σ^n homeomorphic to S^n?

- Yes. (Generalized) Poincaré conjecture
- Smale (1962): $n \geq 5$
- Freedman (1982): $n = 4$
- Perelman (2002): $n = 3$

Smooth structures on S^n

Question

Are they all diffeomorphic to S^n, equipped with the usual smooth structure?

Smooth structures on S^n

Question

Are they all diffeomorphic to S^n, equipped with the usual smooth structure?

- n $=$3: True (Moise's Theorem 1952)
- n $=$ 4: wide open
- n $=$ 7: Milnor constructed an exotic 7-sphere (1956)

Smooth structures on S^n

Question

Are they all diffeomorphic to S^n, equipped with the usual smooth structure?

- n $=$3: True (Moise's Theorem 1952)
- n $=$ 4: wide open
- n $=$ 7: Milnor constructed an exotic 7-sphere (1956)
- Kervaire and Milnor (1963) computed the groups of exotic n-spheres ($n > 4$) in terms of $\pi_n^{st} S^0$, modulo the Kervaire invariant

Kervaire–Milnor

- Θ_n = group of homotopy n-spheres up to diffeomorphism (connected sum)

Kervaire–Milnor

- Θ_n = group of homotopy n-spheres up to diffeomorphism (connected sum)
- Θ_n^{bp} = homotopy n-spheres that bounds parallelizable manifolds

Kervaire–Milnor

- Θ_n = group of homotopy n-spheres up to diffeomorphism (connected sum)
- Θ_n^{bp} = homotopy n-spheres that bounds parallelizable manifolds

Theorem (Kervaire–Milnor)

For $n \geq 5$, the subgroup Θ_n^{bp} is cyclic,

$$|\Theta_n^{bp}| = \begin{cases} 1, & n \text{ even} \\ 1 \text{ or } 2, & n = 4k+1 \\ b_k, & n = 4k-1 \end{cases}$$

$b_k = 2^{2k-2}(2^{2k-1}-1) \cdot$ *numerator of* $\frac{4B_{2k}}{k}$
B_{2k}*: Bernoulli number*

Theorem (Kervaire–Milnor)

- *For* $n \not\equiv 2 \pmod 4$, *there is an exact sequence*

$$0 \longrightarrow \Theta_n^{bp} \longrightarrow \Theta_n \longrightarrow \pi_n/J \longrightarrow 0$$

π_n: *n-th stable homotopy groups of spheres*
π_n/J: *cokernel of the J-homomorphism*

Theorem (Kervaire–Milnor)

- *For* $n \not\equiv 2 \pmod 4$, *there is an exact sequence*

$$0 \longrightarrow \Theta_n^{bp} \longrightarrow \Theta_n \longrightarrow \pi_n/J \longrightarrow 0$$

π_n: *n-th stable homotopy groups of spheres*
π_n/J: *cokernel of the J-homomorphism*

- *For* $n \equiv 2 \pmod 4$, *there is an exact sequence*

$$0 \longrightarrow \Theta_n^{bp} \longrightarrow \Theta_n \longrightarrow \pi_n/J \xrightarrow{\Phi_n} \mathbb{Z}/2 \longrightarrow \Theta_{n-1}^{bp} \longrightarrow 0$$

Φ_n: *the Kervaire invariant*

Theorem (Kervaire–Milnor)

- *For* $n \not\equiv 2 \pmod 4$, *there is an exact sequence*

$$0 \longrightarrow \Theta_n^{bp} \longrightarrow \Theta_n \longrightarrow \pi_n/J \longrightarrow 0$$

  π_n: *n-th stable homotopy groups of spheres*
  π_n/J: *cokernel of the J-homomorphism*

- *For* $n \equiv 2 \pmod 4$, *there is an exact sequence*

$$0 \longrightarrow \Theta_n^{bp} \longrightarrow \Theta_n \longrightarrow \pi_n/J \xrightarrow{\Phi_n} \mathbb{Z}/2 \longrightarrow \Theta_{n-1}^{bp} \longrightarrow 0$$

  Φ_n: *the Kervaire invariant*

The Kervaire invariant problem is the last missing piece of this puzzle

The Kervaire invariant one problem

Question

In which dimensions is there a framed manifold with Kervaire invariant one?

The Kervaire invariant one problem

Question

In which dimensions is there a framed manifold with Kervaire invariant one?

- At the time (1963): 2, 6, ~~10~~, 14, ~~18~~

The Kervaire invariant one problem

Question

In which dimensions is there a framed manifold with Kervaire invariant one?

- At the time (1963): 2, 6, ~~10~~, 14, ~~18~~
- Unclear that it is related to stable homotopy theory

The Kervaire invariant one problem

Question

In which dimensions is there a framed manifold with Kervaire invariant one?

- At the time (1963): 2, 6, ~~10~~, 14, ~~18~~
- Unclear that it is related to stable homotopy theory
- Next piece of the puzzle was unlocked by Browder

Theorem (Browder 1969)

If $\Phi(M) = 1$, *then* $\dim(M) = 2^{j+1} - 2$.

Theorem (Browder 1969)

If $\Phi(M) = 1$*, then* $\dim(M) = 2^{j+1} - 2$*.*
There exists a framed manifold of Kervaire invariant one
$\iff h_j^2 \in Ext_{\mathcal{A}}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ *survives the Adams spectral sequence to an element* $\theta_j \in \pi_{2^{j+1}-2} S^0$

Theorem (Browder 1969)

If $\Phi(M) = 1$, *then* $\dim(M) = 2^{j+1} - 2$.
There exists a framed manifold of Kervaire invariant one $\iff h_j^2 \in Ext_{\mathcal{A}}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ *survives the Adams spectral sequence to an element* $\theta_j \in \pi_{2^{j+1}-2}S^0$

- $h_j \in \mathrm{Ext}_{\mathcal{A}}^{1,2^j}(\mathbb{F}_2, \mathbb{F}_2)$: Hopf invariant one elements, only the first three survives (θ_1, θ_2, θ_3)

Theorem (Browder 1969)

If $\Phi(M) = 1$, *then* $\dim(M) = 2^{j+1} - 2$.
There exists a framed manifold of Kervaire invariant one $\iff h_j^2 \in Ext_{\mathcal{A}}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ *survives the Adams spectral sequence to an element* $\theta_j \in \pi_{2^{j+1}-2} S^0$

- $h_j \in \mathrm{Ext}_{\mathcal{A}}^{1,2^j}(\mathbb{F}_2, \mathbb{F}_2)$: Hopf invariant one elements, only the first three survives (θ_1, θ_2, θ_3)
- Mahowald–Tangora: $\theta_4 \in \pi_{30} S^0$ exists
- Barrat–Jones–Mahowald: $\theta_5 \in \pi_{62} S^0$ exists

Theorem (Browder 1969)

If $\Phi(M) = 1$, then $\dim(M) = 2^{j+1} - 2$.
There exists a framed manifold of Kervaire invariant one $\Longleftrightarrow h_j^2 \in Ext_{\mathcal{A}}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ survives the Adams spectral sequence to an element $\theta_j \in \pi_{2^{j+1}-2} S^0$

- $h_j \in \mathrm{Ext}_{\mathcal{A}}^{1,2^j}(\mathbb{F}_2, \mathbb{F}_2)$: Hopf invariant one elements, only the first three survives (θ_1, θ_2, θ_3)
- Mahowald–Tangora: $\theta_4 \in \pi_{30} S^0$ exists
- Barrat–Jones–Mahowald: $\theta_5 \in \pi_{62} S^0$ exists
- However, the Adams spectral sequence gets very hard at higher dimensions

Theorem (Browder 1969)

If $\Phi(M) = 1$*, then* $\dim(M) = 2^{j+1} - 2$*.*
There exists a framed manifold of Kervaire invariant one
$\iff h_j^2 \in Ext_{\mathcal{A}}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ *survives the Adams spectral sequence to an element* $\theta_j \in \pi_{2^{j+1}-2} S^0$

- $h_j \in \mathrm{Ext}_{\mathcal{A}}^{1,2^j}(\mathbb{F}_2, \mathbb{F}_2)$: Hopf invariant one elements, only the first three survives (θ_1, θ_2, θ_3)
- Mahowald–Tangora: $\theta_4 \in \pi_{30} S^0$ exists
- Barrat–Jones–Mahowald: $\theta_5 \in \pi_{62} S^0$ exists
- However, the Adams spectral sequence gets very hard at higher dimensions
- What about the fate of the higher θ_j's?

Hill–Hopkins–Ravenel Theorems

Theorem (Hill–Hopkins–Ravenel)

For $j \geq 7$, the Kervaire invariant elements θ_j do not exist.

Hill–Hopkins–Ravenel Theorems

Theorem (Hill–Hopkins–Ravenel)

For $j \geq 7$, the Kervaire invariant elements θ_j do not exist.

- The case for $\theta_6 \in \pi_{126} S^0$ is still open

Hill–Hopkins–Ravenel Theorems

Theorem (Hill–Hopkins–Ravenel)

For $j \geq 7$, the Kervaire invariant elements θ_j do not exist.

- The case for $\theta_6 \in \pi_{126} S^0$ is still open
- Start with $MU^{((C_8))}$

Hill–Hopkins–Ravenel Theorems

Theorem (Hill–Hopkins–Ravenel)

For $j \geq 7$, the Kervaire invariant elements θ_j do not exist.

- The case for $\theta_6 \in \pi_{126} S^0$ is still open
- Start with $MU^{((C_8))}$
- Invert a certain class $D \in \pi_{\star}^{C_8} MU^{((C_8))}$: $D^{-1} MU^{((C_8))}$

Hill–Hopkins–Ravenel Theorems

Theorem (Hill–Hopkins–Ravenel)

For $j \geq 7$, the Kervaire invariant elements θ_j do not exist.

- The case for $\theta_6 \in \pi_{126} S^0$ is still open
- Start with $MU^{(\!(C_8)\!)}$
- Invert a certain class $D \in \pi_\star^{C_8} MU^{(\!(C_8)\!)}$: $D^{-1} MU^{(\!(C_8)\!)}$
- Ω: its C_8-fixed point spectrum

Hill–Hopkins–Ravenel Theorems

Theorem (Hill–Hopkins–Ravenel)

For $j \geq 7$, the Kervaire invariant elements θ_j do not exist.

- The case for $\theta_6 \in \pi_{126} S^0$ is still open
- Start with $MU^{((C_8))}$
- Invert a certain class $D \in \pi_\star^{C_8} MU^{((C_8))}$: $D^{-1} MU^{((C_8))}$
- Ω: its C_8-fixed point spectrum
- Detection Theorem:
  If θ_j exists, then its image in $\pi_{2^{j+1}-2}\Omega$ is nonzero

Hill–Hopkins–Ravenel Theorems

Theorem (Hill–Hopkins–Ravenel)

For $j \geq 7$, the Kervaire invariant elements θ_j do not exist.

- The case for $\theta_6 \in \pi_{126} S^0$ is still open
- Start with $MU^{((C_8))}$
- Invert a certain class $D \in \pi_{\bigstar}^{C_8} MU^{((C_8))}$: $D^{-1} MU^{((C_8))}$
- Ω: its C_8-fixed point spectrum
- Detection Theorem:
  If θ_j exists, then its image in $\pi_{2^{j+1}-2}\Omega$ is nonzero
- Periodicity Theorem:
  $\pi_* \Omega$ is 256-periodic

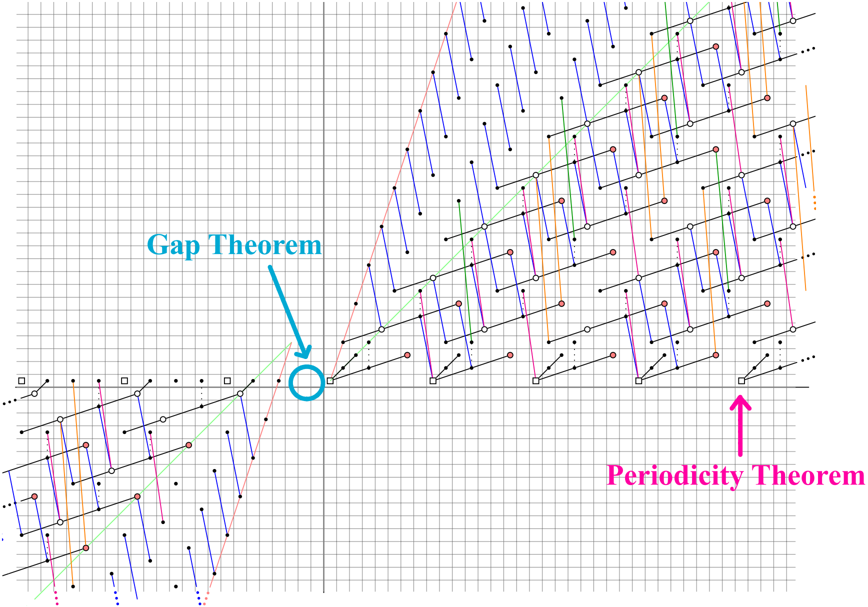
Hill–Hopkins–Ravenel Theorems

Theorem (Hill–Hopkins–Ravenel)

For $j \geq 7$, the Kervaire invariant elements θ_j do not exist.

- The case for $\theta_6 \in \pi_{126} S^0$ is still open
- Start with $MU^{((C_8))}$
- Invert a certain class $D \in \pi_{\star}^{C_8} MU^{((C_8))}$: $D^{-1} MU^{((C_8))}$
- Ω: its C_8-fixed point spectrum
- Detection Theorem:
  If θ_j exists, then its image in $\pi_{2^{j+1}-2}\Omega$ is nonzero
- Periodicity Theorem:
  $\pi_*\Omega$ is 256-periodic
- Gap Theorem:
  $\pi_i\Omega = 0$ for $i = -1, -2, -3$

Baby Ω



$MU^{((G))}$ vs. E_n^{hG}

- Originally, Hill–Hopkins–Ravenel didn't plan to use $MU^{((C_8))}$

$MU^{(\!(G)\!)}$ vs. E_n^{hG}

- Originally, Hill–Hopkins–Ravenel didn't plan to use $MU^{(\!(C_8)\!)}$
- $E_4^{hC_8}$ also detects θ_j

$MU^{((G))}$ vs. E_n^{hG}

- Originally, Hill–Hopkins–Ravenel didn't plan to use $MU^{((C_8))}$
- $E_4^{hC_8}$ also detects θ_j
- However, its HFPSS is HARD!

$MU^{((G))}$ vs. E_n^{hG}

- Originally, Hill–Hopkins–Ravenel didn't plan to use $MU^{((C_8))}$
- $E_4^{hC_8}$ also detects θ_j
- However, its HFPSS is HARD!
- In the end, they settled with $MU^{((C_8))}$

$MU^{(\!(G)\!)}$ vs. E_n^{hG}

$MU^{(\!(G)\!)}$:

- Genuine equivariant homotopy theory (rigid).
- Accessible to computations: SliceSS($MU^{(\!(G)\!)}$).

E_n^{hG}:

-
-
-

$MU^{((G))}$ vs. E_n^{hG}

$MU^{((G))}$:

- Genuine equivariant homotopy theory (rigid).
- Accessible to computations: SliceSS($MU^{((G))}$).

E_n^{hG}:

- Borel equivariant homotopy theory (not rigid).
- HFPSS(E_n^{hG}) is hard.
-

$MU^{((G))}$ vs. E_n^{hG}

$MU^{((G))}$:

- Genuine equivariant homotopy theory (rigid).
- Accessible to computations: SliceSS($MU^{((G))}$).

E_n^{hG}:

- Borel equivariant homotopy theory (not rigid).
- HFPSS(E_n^{hG}) is hard.
- Perfect for doing chromatic homotopy theory.

$MU^{((G))}$ vs. E_n^{hG}

$MU^{((G))}$:

- Genuine equivariant homotopy theory (rigid).
- Accessible to computations: SliceSS($MU^{((G))}$).

E_n^{hG}:

- ~~Borel equivariant homotopy theory (not rigid).~~
- ~~HFPSS(E_n^{hG}) is hard.~~
- Perfect for doing chromatic homotopy theory.

The Real orientation combines the pros and gets rid of the cons!

$MU^{((G))}$ vs. E_n^{hG}

$MU^{((G))}$:

- Genuine equivariant homotopy theory (rigid).
- Accessible to computations: SliceSS($MU^{((G))}$).

E_n^{hG}:

- ~~Borel equivariant homotopy theory (not rigid).~~
- ~~HFPSS(E_n^{hG}) is hard.~~
- Perfect for doing chromatic homotopy theory.

The Real orientation combines the pros and gets rid of the cons!
We can now use the slice spectral sequence to compute E_n^{hG}!

The detection tower

$$\begin{tikzcd}
& (MU^{((C_{2m}))})^{C_{2m}} \arrow[r] \arrow[d] & E_{2^{m-1}n}^{hC_{2m}} \\
& \vdots \arrow[d] & \\
S^0 \arrow[uur] \arrow[r] \arrow[dr] \arrow[ddr] & (MU^{((C_8))})^{C_8} \arrow[r] \arrow[d] & E_{4n}^{hC_8} \\
& (MU^{((C_4))})^{C_4} \arrow[r] \arrow[d] & E_{2n}^{hC_4} \\
& (MU_{\mathbb{R}})^{C_2} \arrow[r] & E_n^{hC_2}
\end{tikzcd}$$

The detection tower

$$
\begin{tikzcd}
 & (MU^{((C_{2^m}))})^{C_{2^m}} \arrow[r] \arrow[d] & E_{2^{m-1}n}^{hC_{2^m}} \\
 & \vdots \arrow[d] & \\
S^0 \arrow[ruu] \arrow[r] \arrow[rd] \arrow[rdd] & (MU^{((C_8))})^{C_8} \arrow[r] \arrow[d] & E_{4n}^{hC_8} \\
 & (MU^{((C_4))})^{C_4} \arrow[r] \arrow[d] & E_{2n}^{hC_4} \\
 & (MU_{\mathbb{R}})^{C_2} \arrow[r] & E_n^{hC_2}
\end{tikzcd}
$$

- As we move up the tower, the Hurewicz images increase and the theories become more powerful

The detection tower

$$
\begin{tikzcd}
 & (MU^{((C_{2^m}))})^{C_{2^m}} \arrow[r] \arrow[d] & E_{2^{m-1}n}^{hC_{2^m}} \\
 & \vdots \arrow[d] & \\
S^0 \arrow[ruu] \arrow[r] \arrow[rd] \arrow[rdd] & (MU^{((C_8))})^{C_8} \arrow[r] \arrow[d] & E_{4n}^{hC_8} \\
 & (MU^{((C_4))})^{C_4} \arrow[r] \arrow[d] & E_{2n}^{hC_4} \\
 & (MU_{\mathbb{R}})^{C_2} \arrow[r] & E_n^{hC_2}
\end{tikzcd}
$$

- As we move up the tower, the Hurewicz images increase and the theories become more powerful
- **Goal: analyze this tower as much as possible**

Classically:

$$\begin{tikzcd}
BP \arrow[r] & \cdots \arrow[r] & BP\langle 3\rangle \arrow[r] \arrow[d] & BP\langle 2\rangle \arrow[r] \arrow[d] & BP\langle 1\rangle \arrow[d] \\
& & v_3^{-1}BP\langle 3\rangle \arrow[d, rightsquigarrow] & v_2^{-1}BP\langle 2\rangle \arrow[d, rightsquigarrow] & v_1^{-1}BP\langle 1\rangle \arrow[d, rightsquigarrow] \\
& & E_3 & E_2 & E_1
\end{tikzcd}$$

- 2-locally, MU splits as a wedge of suspensions of BP

Classically:

$$
\begin{tikzcd}
BP \arrow[r] & \cdots \arrow[r] & BP\langle 3\rangle \arrow[r] \arrow[d] & BP\langle 2\rangle \arrow[r] \arrow[d] & BP\langle 1\rangle \arrow[d] \\
 & & v_3^{-1}BP\langle 3\rangle \arrow[d, rightsquigarrow] & v_2^{-1}BP\langle 2\rangle \arrow[d, rightsquigarrow] & v_1^{-1}BP\langle 1\rangle \arrow[d, rightsquigarrow] \\
 & & E_3 & E_2 & E_1
\end{tikzcd}
$$

- 2-locally, MU splits as a wedge of suspensions of BP
- $\pi_* BP\langle n\rangle = \mathbb{Z}_{(2)}[v_1, \ldots, v_n]$

Classically:

$$
\begin{tikzcd}
BP \arrow[r] & \cdots \arrow[r] & BP\langle 3\rangle \arrow[r] \arrow[d] & BP\langle 2\rangle \arrow[r] \arrow[d] & BP\langle 1\rangle \arrow[d] \\
 & & v_3^{-1}BP\langle 3\rangle \arrow[d, rightsquigarrow] & v_2^{-1}BP\langle 2\rangle \arrow[d, rightsquigarrow] & v_1^{-1}BP\langle 1\rangle \arrow[d, rightsquigarrow] \\
 & & E_3 & E_2 & E_1
\end{tikzcd}
$$

- 2-locally, MU splits as a wedge of suspensions of BP
- $\pi_* BP\langle n\rangle = \mathbb{Z}_{(2)}[v_1, \ldots, v_n]$
- The formal group laws associated with $BP\langle n\rangle$ give models for E_n

Models of Lubin–Tate spectra

$$
\begin{tikzcd}
BP^{((C_{2^m}))} \arrow[r] & BP^{((C_{2^m}))}\langle 3\rangle \arrow[r] \arrow[d] & BP^{((C_{2^m}))}\langle 2\rangle \arrow[r] \arrow[d] & BP^{((C_{2^m}))}\langle 1\rangle \arrow[d] \\
& D_3^{-1}BP^{((C_{2^m}))}\langle 3\rangle \arrow[d, squiggly] & D_2^{-1}BP^{((C_{2^m}))}\langle 2\rangle \arrow[d, squiggly] & D_1^{-1}BP^{((C_{2^m}))}\langle 1\rangle \arrow[d, squiggly] \\
& E_{3\cdot 2^{m-1}} \arrow[loop below, "C_{2^m}"] & E_{2\cdot 2^{m-1}} \arrow[loop below, "C_{2^m}"] & E_{2^{m-1}} \arrow[loop below, "C_{2^m}"]
\end{tikzcd}
$$

Theorem (Beaudry–Hill–S.–Zeng)

The equivariant formal group laws associated with $BP^{((C_{2^m}))}\langle n\rangle$ give good models of $E_{2^{m-1}\cdot n}$, equipped with a C_{2^m}-action.

Models of Lubin–Tate spectra

$$
\begin{tikzcd}
BP^{((C_{2^m}))} \arrow[r] & BP^{((C_{2^m}))}\langle 3\rangle \arrow[r] \arrow[d] & BP^{((C_{2^m}))}\langle 2\rangle \arrow[r] \arrow[d] & BP^{((C_{2^m}))}\langle 1\rangle \arrow[d] \\
 & D_3^{-1}BP^{((C_{2^m}))}\langle 3\rangle \arrow[d, squiggly] & D_2^{-1}BP^{((C_{2^m}))}\langle 2\rangle \arrow[d, squiggly] & D_1^{-1}BP^{((C_{2^m}))}\langle 1\rangle \arrow[d, squiggly] \\
 & E_{3.2^{m-1}} \arrow[loop below, "C_{2^m}"] & E_{2.2^{m-1}} \arrow[loop below, "C_{2^m}"] & E_{2^{m-1}} \arrow[loop below, "C_{2^m}"]
\end{tikzcd}
$$

Theorem (Beaudry–Hill–S.–Zeng)

The equivariant formal group laws associated with $BP^{((C_{2^m}))}\langle n\rangle$ give good models of $E_{2^{m-1}.n}$, equipped with a C_{2^m}-action.

These models are great for doing computations

$BP^{((C_4))}\langle 1\rangle$

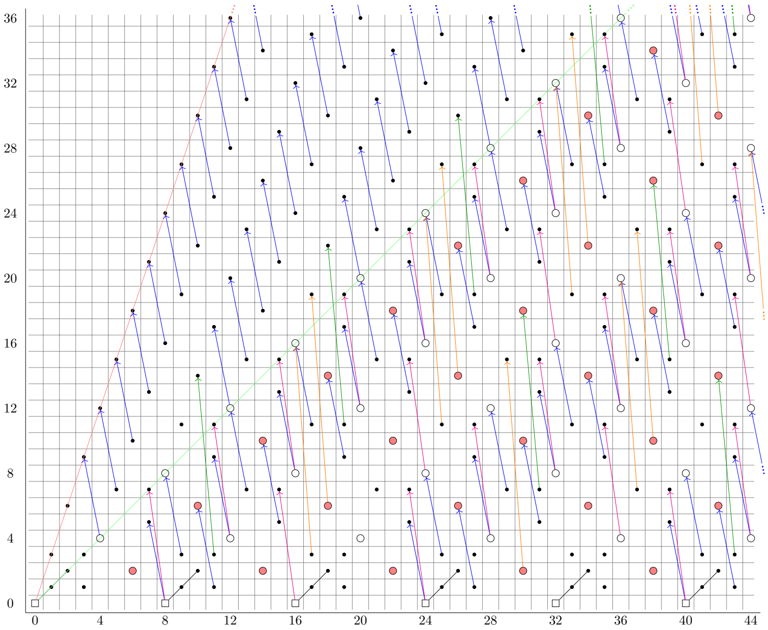
$$\begin{tikzcd}
BP^{((C_4))} \arrow[r] & BP^{((C_4))}\langle 3\rangle \arrow[r] \arrow[d] & BP^{((C_4))}\langle 2\rangle \arrow[r] \arrow[d] & BP^{((C_4))}\langle 1\rangle \arrow[d] \\
 & D_3^{-1}BP^{((C_4))}\langle 3\rangle \arrow[d, rightsquigarrow] & D_2^{-1}BP^{((C_4))}\langle 2\rangle \arrow[d, rightsquigarrow] & D_1^{-1}BP^{((C_4))}\langle 1\rangle \arrow[d, rightsquigarrow] \\
 & E_6 \arrow[loop below, "C_4"] & E_4 \arrow[loop below, "C_4"] & E_2 \arrow[loop below, "C_4"]
\end{tikzcd}$$

$BP^{(\!(C_4)\!)}\langle 1\rangle$

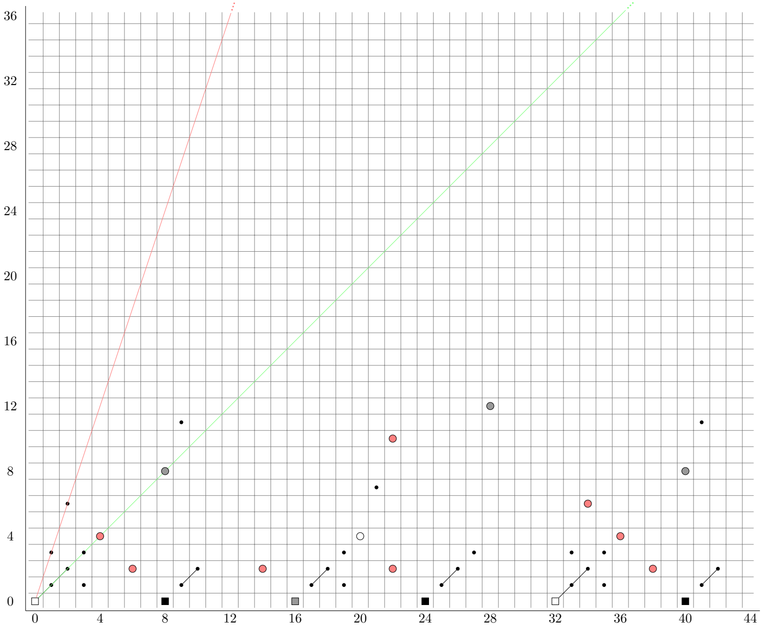
$$\begin{tikzcd}
BP^{(\!(C_4)\!)} \arrow[r] & BP^{(\!(C_4)\!)}\langle 3\rangle \arrow[r] \arrow[d] & BP^{(\!(C_4)\!)}\langle 2\rangle \arrow[r] \arrow[d] & \color{red}{BP^{(\!(C_4)\!)}\langle 1\rangle} \arrow[d, red] \\
 & D_3^{-1}BP^{(\!(C_4)\!)}\langle 3\rangle \arrow[d, rightsquigarrow] & D_2^{-1}BP^{(\!(C_4)\!)}\langle 2\rangle \arrow[d, rightsquigarrow] & \color{red}{D_1^{-1}BP^{(\!(C_4)\!)}\langle 1\rangle} \arrow[d, rightsquigarrow, red] \\
 & E_6 \arrow[loop below, "C_4"] & E_4 \arrow[loop below, "C_4"] & \color{red}{E_2} \arrow[loop below, red, "C_4"]
\end{tikzcd}$$

- $TMF_0(5)$
- Behrens–Ormsby, Hill–Hopkins–Ravenel, Beaudry–Bobkova–Hill–Stojanoska

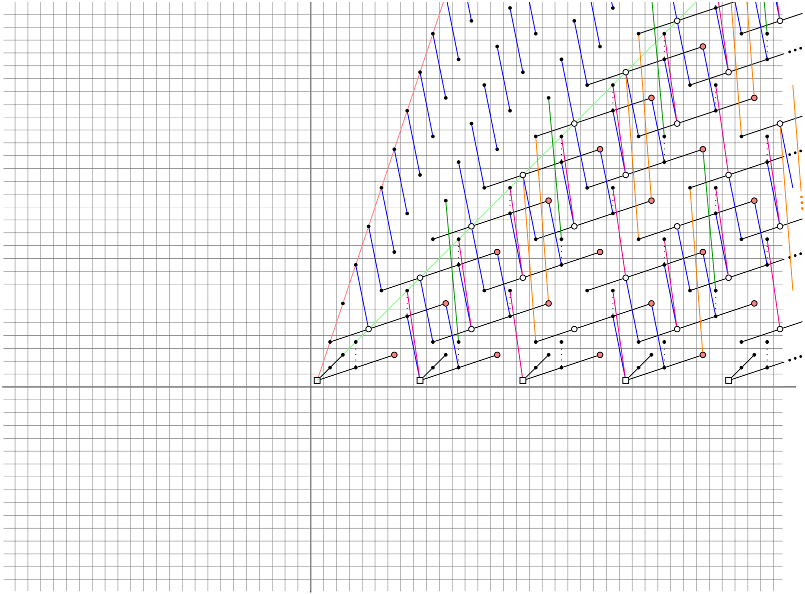
SliceSS$(BP^{((C_4))}\langle 1\rangle)$



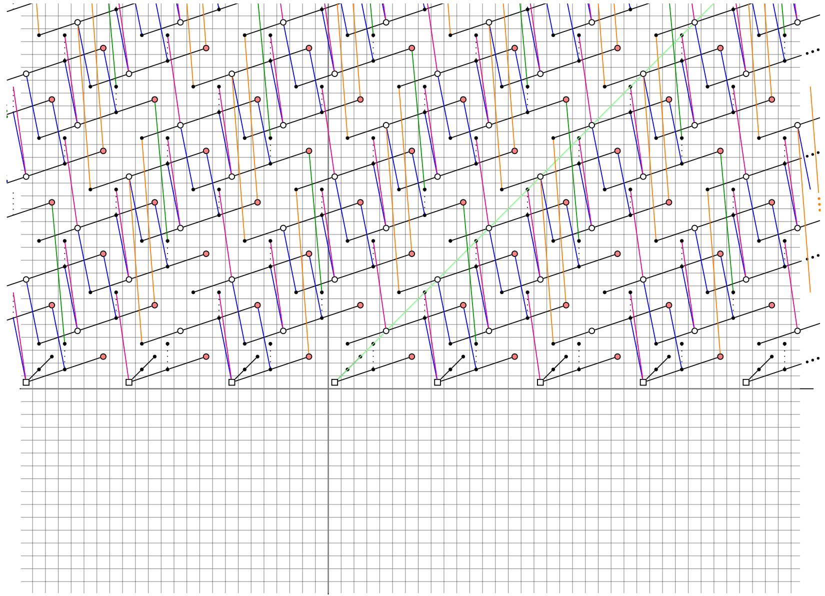
$\mathrm{SliceSS}(BP^{((C_4))}\langle 1\rangle)$



SliceSS$(BP^{(\!(C_4)\!)}\langle 1\rangle)$



HFPSS$(E_2^{hC_4})$



$BP^{(\!(C_4)\!)}\langle 2\rangle$

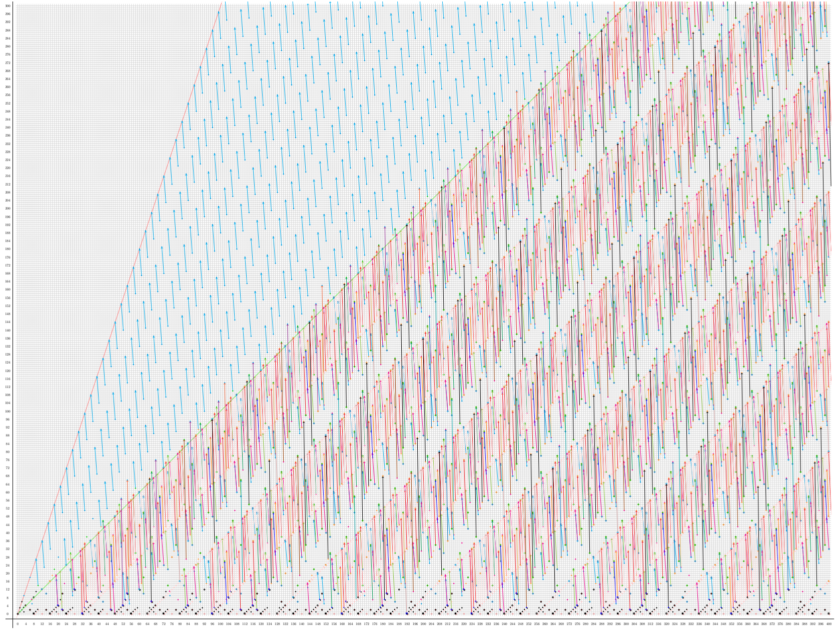
$$\begin{tikzcd}
BP^{(\!(C_4)\!)} \arrow[r] & BP^{(\!(C_4)\!)}\langle 3\rangle \arrow[r] \arrow[d] & BP^{(\!(C_4)\!)}\langle 2\rangle \arrow[r] \arrow[d] & BP^{(\!(C_4)\!)}\langle 1\rangle \arrow[d] \\
 & D_3^{-1}BP^{(\!(C_4)\!)}\langle 3\rangle \arrow[d, squiggly] & D_2^{-1}BP^{(\!(C_4)\!)}\langle 2\rangle \arrow[d, squiggly] & D_1^{-1}BP^{(\!(C_4)\!)}\langle 1\rangle \arrow[d, squiggly] \\
 & E_6 \arrow[loop below, "C_4"] & E_4 \arrow[loop below, "C_4"] & E_2 \arrow[loop below, "C_4"]
\end{tikzcd}$$

$BP^{((C_4))}\langle 2\rangle$

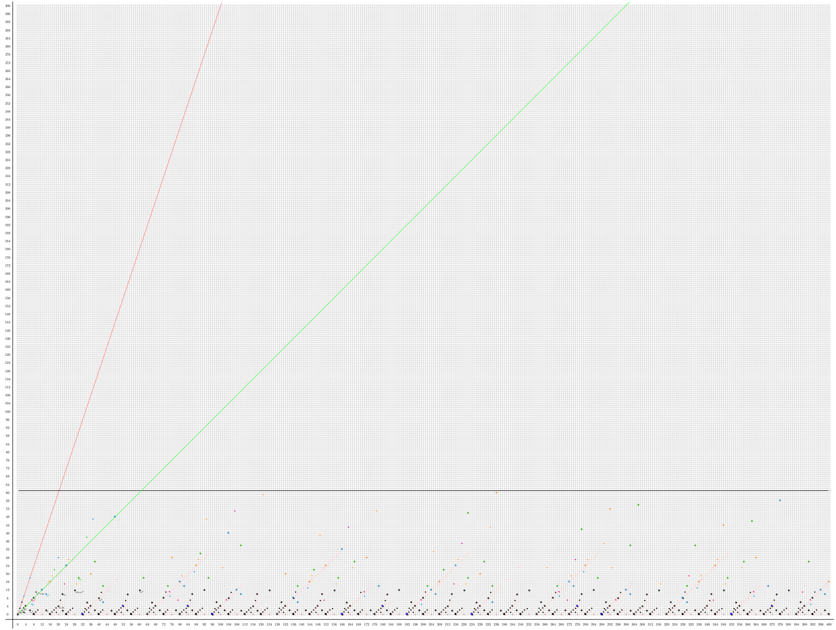
$$
\begin{tikzcd}
BP^{((C_4))} \arrow[r] & BP^{((C_4))}\langle 3\rangle \arrow[r] \arrow[d] & BP^{((C_4))}\langle 2\rangle \arrow[r] \arrow[d] & BP^{((C_4))}\langle 1\rangle \arrow[d] \\
& D_3^{-1}BP^{((C_4))}\langle 3\rangle \arrow[d, squiggly] & D_2^{-1}BP^{((C_4))}\langle 2\rangle \arrow[d, squiggly] & D_1^{-1}BP^{((C_4))}\langle 1\rangle \arrow[d, squiggly] \\
& E_6 \arrow[loop below, "C_4"] & E_4 \arrow[loop below, "C_4"] & E_2 \arrow[loop below, "C_4"]
\end{tikzcd}
$$

- Hill–S.–Wang–Xu
- First height > 2 computation where the group is bigger than C_2

SliceSS($BP^{((C_4))}\langle 2\rangle$)



$\mathrm{SliceSS}(BP^{((C_4))}\langle 2\rangle) : E_\infty$



Periodicity Theorem

- This is the first height > 2 computation where the group is bigger than C_2

Periodicity Theorem

- This is the first height > 2 computation where the group is bigger than C_2
- Our computation implies that $E_4^{hC_{12}}$ is 384-periodic

Periodicity Theorem

- This is the first height > 2 computation where the group is bigger than C_2
- Our computation implies that $E_4^{hC_{12}}$ is 384-periodic
- Question: what is the period of E_n^{hG} for finite $G \subset \mathbb{G}_n$?

Periodicity Theorem

- This is the first height > 2 computation where the group is bigger than C_2
- Our computation implies that $E_4^{hC_{12}}$ is 384-periodic
- Question: what is the period of E_n^{hG} for finite $G \subset \mathbb{G}_n$?

Theorem (Beaudry–Hill–S.–Wang–Xu–Zeng, Periodicity Theorem)

1. *The spectrum $E_{n\cdot 2^{m-1}}^{hC_{2^m}}$ is periodic with period $2^{n\cdot 2^{m-1}+m+1}$.*
2. *The spectrum $E_{4n+2}^{hQ_8}$ is periodic with period 2^{4n+6}.*

Periodicity Theorem

- This is the first height > 2 computation where the group is bigger than C_2
- Our computation implies that $E_4^{hC_{12}}$ is 384-periodic
- Question: what is the period of E_n^{hG} for finite $G \subset \mathbb{G}_n$?

Theorem (Beaudry–Hill–S.–Wang–Xu–Zeng, Periodicity Theorem)

1. *The spectrum $E_{n\cdot 2^{m-1}}^{hC_{2^m}}$ is periodic with period $2^{n\cdot 2^{m-1}+m+1}$.*
2. *The spectrum $E_{4n+2}^{hQ_8}$ is periodic with period 2^{4n+6}.*

- This resolves the periodicity of E_n^{hG} at all heights and all G


Thank you!