Perspectives on Nonabelian Hodge theory

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Classical Hodge theory

Let (X, g) be a Riemannian manifold Let

$$\Omega_X^\bullet = C^\infty(X) \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \dots$$

The presence of a metric allows for one to define an adjoint operator $\delta : \Omega^\bullet \to \Omega^{\bullet-1}$

Definition
Let $\Delta = \delta d + d\delta$. This is the *Laplacian*. A differential n-form is *harmonic* if $\Delta\omega = 0$.

Theorem (Hodge theorem)
There is an isomorphism

$$H^n(X, \mathbb{R}) \cong H^n_{dR}(X) \cong \mathcal{H}^n(X),$$

where the right hand side denotes the real vector space of harmonic n-forms.

Classical Hodge theory: Kähler case

Remark
One therefore has harmonic representatives for de Rham cohomology classes.

Now let's assume that X is a smooth projective complex variety (more generally one may assume X is Kähler but in this talk we can just take that to mean the above.)

The complex structure on the tangent bundle gives rise to a decomposition of complexified differential forms

$$\Omega^n_X \otimes \mathbb{C} = \bigoplus_{p+q=n} \Omega^{p,q}_X$$

with $\Omega^{p,q} = (\Omega^{1,0}_X \wedge \dots \wedge \Omega^{1,0}_X) \wedge (\Omega^{0,1}_X \wedge \dots \wedge \Omega^{0,1}_X)$

In local holomorphic coordinates $(z_1, \dots z_j)$, $\Omega^{1,0}_X$ consists of forms $\omega = \sum c_j dz_j$; meanwhile $\Omega^{0,1}$ consists of forms $\omega = \sum c_j d\overline{z}_j$

Additionally the de Rham differential decomposes as

$$d = \partial + \overline{\partial}$$

with $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$

Definition
One has a refined notion of harmonic forms. There will be formal adjoints $\partial^*, \overline{\partial}^*$. One defines Δ_d, Δ_∂ and $\Delta_{\overline{\partial}}$ analogously to above.

The *Kahler* identities state that $\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$, hence the condition of being harmonic is equivalent.

The Hodge theorem

Theorem
There exists a decomposiion

$$H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}(X)$$

where $H^{p,q}(X) = H^p(X, \Omega_X^q) = \mathcal{H}^{p,q}(X)$*. The latter is the vector space of harmonic forms of degree (p,q).*

Remark
This data gives rise to the notion of a "pure Hodge structure of weight n". This is equivalent to a finite decreasing filtration on a complex vector space $F^{p+1} \subset F^p \subset$ ...together with a conjugate filtration s.t. $F^p(H) \cap \overline{F^q(H)} = H^{p,q}$

The nonabelian Hodge correspondence

The striking fact here is that we are mixing data from topological setting ($H^n(X, \mathbb{C})$), the smooth setting $H^n_{dR}(X)$ and the decomposition into harmonic forms, which uses the holomorphic (or complex algebro-geometric) structure.

Nonabelian Hodge theory can be viewed as a categorification of this interplay of ideas. To see what we mean by this let's first recall a (rudimentary) version of the Riemann-Hilbert correspondence:

Theorem

There is an equivalence of categories:

$\{$*Local systems of complex vector spaces on* $X\ \} \simeq \{$ *complex vector bundles on* X *with a flat connection* $\}$

Recall, the left hand side can be thought of as representations $\rho : \pi_1(X) \to GL_n(\mathbb{C})$ and the data of the right hand side is pair (E, ∇), with a connection $\nabla : E \to E \otimes \Omega^1_X$ such that $\nabla^2 = 0$ and

$$\nabla(rs) = sd(r) + \nabla(s)r \qquad \textit{Leibniz rule}$$

Riemann-Hilbert

Thus we are led to thinking about complex $\pi_1(X)$-representations, and *bundles with flat connection*. In fact, this motivates our notion of nonabelian cohomology "$H^1(X, GL_n(\mathbb{C}))$". This can be thought of as the space of representations $\rho : \pi_1(X) \to GL_n(\mathbb{C})$ (we'll be a bit more precise later)

How to make sense of the decomposition into $H^p(X, \Omega^q_X)$?

Answer: Vector bundle with "Higgs field" This is a pair (E, θ), where E is a holomorphic bundle, and $\theta : E \to E \otimes \Omega^1_X$ is an $\mathcal{O}_X$-linear map with $\theta \wedge \theta = 0$.

Remark
This structure arose first in the work of Hitchin, in studying self-duality equations on Riemann surface, motivated by ideas from particle physics (analogous version of Higgs fields describe the Higgs boson)

Higgs bundles

One has a notion of harmonic bundle which interpolates between Higgs bundles and flat bundles

Definition
A harmonic bundle on X is a smooth complex vector bundle E with differential operators ∂ and $\overline{\partial}$ along with algebraic (or holomorphic) operators $\theta, \overline{\theta} \in H^0(X, \mathrm{End}(E) \otimes \Omega^1)$.

Construction
One can fix a (Hermitian metric) so that $\partial + \overline{\partial}$ is a unitary connection and $\theta + \overline{\theta}$ is self adjoint. Next, one sets $D = \partial + \overline{\partial} + \theta + \overline{\theta}$ and $D'' = \overline{\partial} + \theta$.
With the above conditions, (E, D) is a vector bundle with flat connection and $(E, \overline{\partial}, \theta) = (E, D'')$ is a Higgs bundle with $\theta \wedge \theta = 0$.

Remark

- The operator $\overline{\partial}$ defines a holomorphic structure on the complex bundle E (follows by the Koszul-Malgrange theorem) so that we do get a holomorphic bundle.
- Given a bundle with flat connection D one uses the presence of the Hermitian metric K to decompose $D = \partial + \overline{\partial} + \theta + \overline{\theta}$. Out of this one builds $D^{"}_K = \theta + \overline{\partial}$ which is the needed data for the Higgs structure. One then needs to solve $(D^{"}_K)^2 = 0$.
- In the reverse direction, given a Higgs bundle, one again uses the metric to define $D_K = \partial_K + \overline{\theta_K}$. Finally one sets $D = D' + D^{"}_K = \partial + \overline{\partial} + \theta + \overline{\theta}$. In order to obtain a bundle with flat connection, one must show $(D_K)^2 = 0$.

The Nonabelian Hodge theorem

The celebrated Hodge theorem states the following:

Theorem (Simpson)
Let X be a smooth complex projective variety. Then:

- *there is a natural equivalence between categories of harmonic bundles on X and semisimple flat bundles (or reps of $\pi_1(X)$).*
- *There is a natural equivalence between categories of harmonic bundles and direct sums of stable Higgs bundles with vanishing Chern class.*
- *The resulting correspondence between representations and Higgs bundles can be extended to an equivalence between the categories of all $\pi_1(X)$ representations and semistable bundles with vanishing Chern class.*

Remark
(Semi-)Stability here means that for every coherent subsheaf $F \subset E$, $\deg(F)/\operatorname{rk}(F) < \deg(E)/\operatorname{rk}(E)$. (resp. $\leq$) This, along with the vanishing of ch_1 and ch_2 is precisely the condition needed for $D^2 = (\partial + \overline{\partial} + \theta + \overline{\theta})^2$ to vanish. In the other direction the associated tensor vanishes only if the bundle with flat connection is semi-simple,

Remark
Some history] The first part of the theorem is due to Corlette and Donaldson, using the work of Eells and Sampson. The second part is a generalization of the theorem of Narasimhan and Seshadri (when $\theta = 0$, following work of Uhlenbeck-Yau and Hitchin and Beilinson-Deligne (and others...)

Categorification?

How can one view this in the light of the classical Hodge correspondence?

In the classical case we have the decomposition

$$H^1(X,\mathbb{C}) \cong H^0(X,\Omega^1) \oplus H^1(X,\mathcal{O}_X).$$

Instead now, one studies the nonabelian cohomology set "$H^1(X, GL_n(\mathbb{C})$" (which as we shall see is really to be thought of as an moduli space/stack...)

$$H^1(X, GL_n(\mathbb{C})) = H^1(X, GL_n(\mathcal{O}_X) \oplus H^0(X, End(E) \otimes \Omega^1_X)$$

The holomorphic bundle is specified by a cocycle corresponding to the first summand, and the operator θ is an element of $H^0(X, End(E) \otimes \Omega^1_X)$.

Consequences on cohomology

Let (V, D) be a bundle with flat connection. To this, one can associate the de Rham complex:

$$A^{\bullet}_{dR}(V) = A^0(V) \xrightarrow{D} A^1(V) \xrightarrow{D} A^2(V) \to \dots$$

where $A^n(V)$ is the global sections of the smooth bundle $V \otimes \Omega_X$.

Meanwhile, let $(E, \theta, \overline{\partial})$ be a Higgs bundle. To this one associates the Dolbeaut complex:

$$A^{\bullet}_{Dol}(E) = A^0(E) \xrightarrow{D"} A^1(E) \xrightarrow{D"} A^2(E) \to \dots$$

(where $D" = \theta + \overline{\partial}$)

Cohomology

Proposition
Suppose $(V, D, D")$ *is a harmonic bundle (so that it corresponds to a flat bundle and a Higgs bundle). Then there are equivalences*

$$A^\bullet_{dR}(V) \simeq (\ker(D), D") \simeq (H_{dR}(V), 0)$$

$$A^\bullet_{Dol}(V) \simeq (\ker(D), D") \simeq (H_{Dol}(V), 0)$$

Remark
The above can be viewed as a formality result for the relevant complexes, which arises from the harmonic structure.

Moduli spaces; algebraic geometry

A key aspect of the nonabelian Hodge correspondence lies in the correspondence of associated moduli spaces.

Let's recall what this is exactly. Let $F : \mathrm{CAlg}_k^0 \to \mathrm{Gpd}$ be a functor from the category of discrete commutative k-algebras to the category of groupoids. Then this functor is representable by scheme/space/stack if there exists some scheme scheme/space/stack X for which $Hom_C(\mathrm{Spec}(A), X) \simeq F(A)$. These are typically required to satisfy some gluing condition (*descent*).
Basic motivation for stacks: the functors are valued in groupoids (eventually we would like them to be valued in ∞-groupoids, i.e. the slogan "points have non-trivial automorphisms...)

More algebraic geometry

Example
Quotient stacks: X a scheme with action by a group scheme G. We define X/G as the realization of the groupoid (ultimately we might as well say "simplicial object")

$$X \leftleftarrows X \times G ...$$

Remark
Other (more specific) notions of quotients are relevant here. There exist a notion of GIT quotient (geometric invariant theory) written $X//G$. IN nice enough cases, this exists as an object in the original category (schemes or algebraic spaces). Locally, this is of the form $(\mathrm{Spec}(R^G))$. It is the quotient of the "semi-stable-points". Often suffices for the moduli problems related to non-Abelian Hodge theory.

Moduli space

Construction (Betti moduli space)
Let Γ *be a finitely generated group. One defines representing scheme paramertizing representations* $R(\Gamma, n) = \mathrm{Map}(\Gamma, GL_n)$ *sending*

$$A \mapsto Hom(\Gamma, GL_n(A))$$

The reductive group GL_n *acts on* $R(\Gamma, n)$ *and we let*

$$\mathcal{M}(\Gamma, GL_n) := R(\Gamma, n)/GL_n$$

denote the quotient stack. Now set, $\mathcal{M}_B(X, n)$ *for* $\Gamma = \pi_1(X^{an})$*.*

Moduli spaces cont.

Construction (de Rham Moduli space)
Fix a point $x \in X$. One defines the de Rham moduli scheme, $R_{dR}(X, n)$ which assigns to a scheme Y the set of isomorphism classes of vector bundles of rank n with flat connection (V, D) and a "frame" $\alpha : V|_x \cong \mathbb{C}^n$. Once again, this admits an action by the algebraic group GL_n and one defines the quotient stack $\mathcal{M}_{dR}(X, n) = R_{dR}(X, n)/GL_n$

Construction (Higgs moduli space)
Finally, one constructs the moduli space of Higgs bundles. One denotes $R_{Dol}(X, x, n)$ to be the moduli scheme of semistable Higgs bundles with vanishing Chern classes and frame at x. Again, this has a GL_n action, with respect to which we take the quotient stack

$$R_{Dol}(X, x, n)/GL_n = \mathcal{M}_{Dol}(X, n)$$

Remark
In fact, originally the moduli spaces that were taken were not the quotient stacks; they were constructed using GIT. These are algebaic spaces. These moduli spaces are the associated coarse moduli spaces for the relevant quotient stacks (they represent the functors π_0 of the stacks). The points of these moduli spaces parametrize semi-simple objects

Moduli Spaces cont.

The identifications at the level of moduli spaces/stacks are rather subtle. First, as a consequence of the Riemann-Hilbert correspondence, one has:

Proposition
There are isomorphisms of associated complex analytic spaces and the analytifications of the moduli spaces (stacks):

$$R_{dR}(X,n)^{an} \simeq R_B(X,n)^{an}, \;\; \mathcal{M}_{dR}(X,n)^{an} \simeq \mathcal{M}_B(X,n)^{an}$$

Theorem
The correspondence provides an isomorphisms between underlying sets of points of $M_{dR}(X,n)$ and $M_{Dol}(X,n)$, as the points of the spaces. This in particular gives homeomorphisms of underlying topological spaces $M_{DR}(X,n)^{top} \simeq M_{Dol}(X,n)^{top}$.

The $\mathbb{C}^*$-action

A particularly interesting facet of the theory is the $\mathbb{C}^*$-action that arises on the category of Higgs bundles.

Construction
Define a $\mathbb{C}^*$ action on the category of semistable Higgs bundles with vanishing Chern class

$$(E, \theta) \mapsto (E, t\theta)$$

Via the equivalence of categories, we can transport this to an action on semi-simple flat bundles.

As a consequence, the the moduli space $\mathcal{M}_{Dol}(X, n)$ acquires a $\mathbb{G}_m$-action.

Some consequences and conjectures

Proposition
Semisimple flat bundles fixed by the action of $\mathbb{C}^* = \mathbb{G}_m$ are exactly those which underlie complex variations of Hodge structure.

Conjecture
A representation ρ of π_1 is rigid if any nearby representation (in $R_B(X, n)$) is conjugate to it. Simpson's motivicity conjecture states that ρ is a direct factor in the monodromy representation of a motive (i.e a family of varieties over X)

Recent work of Esnault-Groechenig making progress towards this conjecture, by finiding a model over integers, and then studying the p-curvatures of the relevant connections modulo p. Utilize characteristic p versions of NAH due to Ogus-Vologodsky, and more recent works of Lan-Sheng-Zuo in the p-adic setting.