Defining limits in the ∞-world

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What is a higher category?

Category

objects

morphisms

+ composition

strictly associative

↳ enriched in Set

↳ internal to Set

2-Category

objects $\bullet$

morphisms $\bullet \longrightarrow \bullet$

2-morphisms

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

+ compositions

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

strictly associative

↳ enriched in Cat

↳ internal to Cat with discrete category of objects.

⇝ Often too strict to accomodate many examples that appear in nature

(∞, n)-categories

objects
morphisms
⋮
n-morphisms
(n+1)-morphisms } invertible
⋮

} $(\infty, n-1)$-category
⋮

+ compositions
associative up to higher invertible morphisms

↳ enriched in $(\infty, n-1)$-categories

↳ internal to $(\infty, n-1)$-categories with discrete $(\infty, n-1)$-category of objects

⇝ To define (∞, n)-categories, we need models

Models of (∞,n)-categories (In this talk) $n=0,1,2$

1) Category $\mathcal{M}$ with a model structure $(\mathcal{M})_{(\infty,n)\text{-Cat}}$ described by its fibrant objects ($=(\infty,n)$-category)

2) Quillen equivalence to an admitted model of (∞,n)-categories

3) Embedding $n\mathrm{Cat} \hookrightarrow (\mathcal{M})_{(\infty,n)\text{-Cat}}$ sending n-categories to (∞,n)-categories

<u>Ex</u>: $n=0$: ∞-groupoids = topological spaces

<u>Kan complex</u>: $X \in \mathrm{sSet}\,(=\mathrm{Set}^{\Delta^{op}})$ st

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \quad \forall\, 0 \le k \le n,\ n \ge 1$$

1) $(\mathrm{sSet})_{\mathrm{Kan}}$ [Quillen]

2) $\mathrm{QE\,Top} \underset{\perp}{\overset{|-|}{\rightleftarrows}} (\mathrm{sSet})_{\mathrm{Kan}}$

3) $\mathrm{Set} \hookrightarrow (\mathrm{sSet})_{\mathrm{Kan}}$

Models of $(\infty,1)$-categories

$(\infty,1)$-categories = enriched in ∞-groupoids

<u>Kan-enriched category</u>: sSet-enriched category $\mathcal{K}$ st the homs $\mathrm{Hom}_{\mathcal{K}}(A,B) \in \mathrm{sSet}$ are Kan complexes $\forall A,B \in \mathcal{K}$

1) $(\mathrm{sSet}\text{-}\mathrm{Cat})_{\mathrm{Kan}}$ [Bergner]

3) $\mathrm{Cat} \hookrightarrow (\mathrm{sSet}\text{-}\mathrm{Cat})_{\mathrm{Kan}}$

<u>Quasi-category</u>: $X \in \mathrm{sSet}$ st

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \quad \forall\, 0<k<n,\ n \geq 1$$

$X_0 = \{\text{objects}\}$; $X_1 = \{\text{morphisms}\}$; composition:

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

1) $(\mathrm{sSet})_{\mathrm{qcat}}$ [Joyal]

2) QE $(\mathrm{sSet}\text{-}\mathrm{Cat})_{\mathrm{Kan}} \underset{\mathcal{N}}{\overset{\mathfrak{C}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} (\mathrm{sSet})_{\mathrm{qcat}}$ [Joyal, Lurie]

3) $N : \mathrm{Cat} \longrightarrow (\mathrm{sSet})_{\mathrm{qcat}}$

$(N\mathcal{C})_n = \mathrm{Cat}([n], \mathcal{C})$

$(\infty,1)$-categories = internal to ∞-groupoids with discrete ∞-groupoid of objects

<u>Complete Segal space</u>: simplicial space $W : \Delta^{op} \to (sSet)_{Kan}$ st

(i) <u>Segal</u>: $W_n \xrightarrow{\sim} W_1 \times_{W_0} \cdots \times_{W_0} W_1$ is a w.e. in $(sSet)_{Kan}$

W_0 = space of objects, W_1 = space of morphisms

Composition: $W_1 \times_{W_0} W_1 \xleftarrow{\sim} W_2 \xrightarrow{d_1} W_1$

$\rightsquigarrow$ underlying ∞-groupoid $W_1^{heq} \subseteq W_1$ of invertible morphisms

(ii) <u>Completeness</u>: $W_0 \xrightarrow{\sim} W_1^{heq}$ is a w.e. in $(sSet)_{Kan}$

1) $(sSet^{\Delta^{op}})_{CSS}$ [Rezk]

2) $\pi_1 : \Delta \times \Delta \to \Delta$, $([n],[k]) \mapsto [n]$; $\iota : \Delta \to \Delta \times \Delta$, $[n] \mapsto ([n],[0])$

$\rightsquigarrow$ QE $(sSet^{\Delta^{op}})_{CSS} \underset{\iota_1^*}{\overset{\pi_1^*}{\rightleftarrows}} (sSet)_{qcat}$ [Joyal-Tierney]

3) $N_R : Cat \longrightarrow (sSet^{\Delta^{op}})_{CSS}$

$(N_R \mathcal{C})_{n,k} = Cat([n] \times I[k], \mathcal{C})$

Limit in a category

$F: I \to \mathcal{C}$ - functor

<u>Limit of F:</u> (L, λ) of an object $L \in \mathcal{C}$ and a cone $\lambda: \Delta L \Rightarrow F$ satisfying one of TFE:

(i) $\lambda_*: \mathcal{C}(X, L) \xrightarrow{\cong} \mathcal{C}^I(\Delta X, F) \quad \forall X \in \mathcal{C}$

(ii) (L, λ) is a terminal object in the category of cones $\Delta \downarrow F$

$$\begin{tikzcd}
\Delta \downarrow F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{I \times [1]} \arrow[d] \\
\mathcal{C} \arrow[r, "(\Delta, \{F\})"'] & \mathcal{C}^I \times \mathcal{C}^I
\end{tikzcd}$$

<u>Terminal object:</u> $T \in \mathcal{C}$ satisfying one of TFE:

(i) $\mathcal{C}(X, T) \cong \{*\} \quad \forall X \in \mathcal{C}$

(ii) $\mathcal{C}_{/T} \xrightarrow{\cong} \mathcal{C}$ is an isomorphism of categories

$$\begin{tikzcd}
\mathcal{C}_{/T} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{[1]} \arrow[d] \\
\mathcal{C} \arrow[r, "(id_{\mathcal{C}}, \{T\})"'] & \mathcal{C} \times \mathcal{C}
\end{tikzcd}$$

Limits in an $(\infty,1)$-category

Enriched version

$F: I \to \mathcal{K} \in \mathrm{sSet}\text{-}\mathrm{Cat}$ with $\mathcal{K}$ Kan-enriched

<u>Homotopy limit of F</u>: (L,λ) of $L \in \mathcal{K}$ and $\lambda: \Delta L \Rightarrow F$ st

$$\lambda_*: \mathrm{Hom}_{\mathcal{K}}(X,L) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}^I}(\Delta X, F)$$

is a w.e. in $(\mathrm{sSet})_{\mathrm{Kan}}$

$\rightsquigarrow$ Retrieves (i) above through the inclusion

$$\mathrm{Cat} \hookrightarrow (\mathrm{sSet}\text{-}\mathrm{Cat})_{\mathrm{Kan}}$$

Terminal version

X – quasi-cat [Joyal]

$x \in X_0$ is a terminal object if $X_{/x} \xrightarrow{\sim} X$ is a w.e. in $(sSet)_{qcat}$

$$\begin{tikzcd} X_{/x} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X^{N[1]} \arrow[d, two heads] \\ X \arrow[r, "(id_X, \{x\})"'] & X \times X \end{tikzcd}$$

$f : K \to X \in sSet$

A limit of f is a terminal object in $\Delta \downarrow f$

$$\begin{tikzcd} \Delta \downarrow f \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X^{K \times N_R[1]} \arrow[d, two heads] \\ X \arrow[r, "(\Delta, \{f\})"'] & X^K \times X^K \end{tikzcd}$$

$\rightsquigarrow$ Retrieves (ii) for Cat through the nerves

W – complete Segal sp. [Rasekh]

$x \in W_{0,0}$ is a terminal object if $W_{/x} \xrightarrow{\sim} W$ is a w.e. in $(sSet^{\Delta^{op}})_{CSS}$

$$\begin{tikzcd} W_{/x} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & W^{N_R[1]} \arrow[d, two heads] \\ W \arrow[r, "(id_W, \{x\})"'] & W \times W \end{tikzcd}$$

$f : K \to W \in sSet^{\Delta^{op}}$

A limit of f is a terminal object in $\Delta \downarrow f$

$$\begin{tikzcd} \Delta \downarrow f \arrow[r] \arrow[d, two heads] & W^{K \times N_R[1]} \arrow[d, two heads] \\ W \arrow[r, "(\Delta, \{f\})"'] & W^K \times W^K \end{tikzcd}$$

Comparison between models

<u>Recall</u>: Quillen equivalences

$$(\mathrm{sSet}\text{-}\mathrm{Cat})_{\mathrm{Kan}} \underset{N}{\overset{\mathfrak{C}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} (\mathrm{sSet})_{\mathrm{qCat}} \underset{\iota^*}{\overset{\pi_1^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathrm{sSet}^{\Delta^{op}})_{\mathrm{CSS}}$$

$\rightsquigarrow$ X-quasi-category

$\exists$ $\mathcal{K}$-Kan-enriched category & W-complete Segal space

$$NJK \xrightarrow{\sim} X \xleftarrow{\sim} \iota^* W$$

<u>Theorem</u>: $K \in \mathrm{sSet}$

(i) [Lurie, Riehl-Verity]

Limit of $f: K \to NJK$ in $(\mathrm{sSet})_{\mathrm{qCat}}$ $\leftrightsquigarrow$ Homotopy limit of $f^{\natural}: \mathfrak{C}K \to \mathcal{K}$ in $(\mathrm{sSet}\text{-}\mathrm{Cat})_{\mathrm{Kan}}$

(ii) [Rasekh, Riehl-Verity]

Limit of $f: K \to \iota^* W$ in $(\mathrm{sSet})_{\mathrm{qCat}}$ $\leftrightsquigarrow$ Limit of $f^{\natural}: \pi_1^* K \to W$ in $(\mathrm{sSet}^{\Delta^{op}})_{\mathrm{CSS}}$

Models of $(\infty,2)$-categories

$(\infty,2)$-category = enriched in $(\infty,1)$-categories

<u>qCat-enriched categories</u>: sSet-enriched category $\mathcal{K}$ st the homs $\mathrm{Hom}_{\mathcal{K}}(A,B) \in \mathrm{sSet}$ are quasi-categories $\forall A,B \in \mathcal{K}$

<u>CSS-enriched categories</u>: $\mathrm{sSet}^{\Delta^{op}}$-enriched category $\mathcal{L}$ st the homs $\mathrm{Hom}_{\mathcal{L}}(A,B) \in \mathrm{sSet}^{\Delta^{op}}$ are complete Segal spaces $\forall A,B \in \mathcal{L}$

1) $(\mathrm{sSet}\text{-}\mathrm{Cat})_{\mathrm{qCat}}$; $(\mathrm{sSet}^{\Delta^{op}}\text{-}\mathrm{Cat})_{\mathrm{CSS}}$ [Lurie]

2) <u>Recall</u>: QE $(\mathrm{sSet}^{\Delta^{op}})_{\mathrm{CSS}} \underset{\iota_1^*}{\overset{\pi_1^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} (\mathrm{sSet})_{\mathrm{qCat}}$

$\rightsquigarrow$ QE $(\mathrm{sSet}^{\Delta^{op}}\text{-}\mathrm{Cat})_{\mathrm{CSS}} \underset{(\iota_1^*)_*}{\overset{(\pi_1^*)_*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} (\mathrm{sSet}\text{-}\mathrm{Cat})_{\mathrm{qCat}}$

3) <u>Recall</u>: $N : \mathrm{Cat} \longrightarrow (\mathrm{sSet})_{\mathrm{qCat}}$, $N_R : \mathrm{Cat} \longrightarrow (\mathrm{sSet}^{\Delta^{op}})_{\mathrm{CSS}}$

$\rightsquigarrow$ $N_* : 2\mathrm{Cat}\,(=\mathrm{Cat}\text{-}\mathrm{Cat}) \longrightarrow (\mathrm{sSet}\text{-}\mathrm{Cat})_{\mathrm{qCat}}$

$(N_R)_* : 2\mathrm{Cat} \longrightarrow (\mathrm{sSet}^{\Delta^{op}}\text{-}\mathrm{Cat})_{\mathrm{CSS}}$

Analogue of complete Segal space

<u>Recall</u>: A complete Segal space is a functor $\Delta^{op} \to sSet$

Category $\Delta \underset{full}{\subseteq} Cat$ with objects

$$[n] = \{0 \to 1 \to \cdots n-1 \to n\}$$

$\rightsquigarrow$ Category $\Theta_2 \underset{full}{\subseteq} 2Cat$ with objects $\quad (\Theta_2 = \Delta \int \Delta)$

$$[n; m_1, \dots, m_n] = \left\{ \begin{tikzcd} 0 \arrow[r, bend left=60, "0"] \arrow[r, bend right=60, "m_1"'] \arrow[r, phantom, "\Downarrow"] & 1 \arrow[r, phantom, "\cdots"] & n-1 \arrow[r, bend left=60, "0"] \arrow[r, bend right=60, "m_n"'] \arrow[r, phantom, "\Downarrow"] & n \end{tikzcd} \right\}$$

<u>Examples</u>: $[0] = \{0\}$; $[1;0] = \{0 \to 1\}$; $[1;1] = \{0 \overset{\Downarrow}{\rightrightarrows} 1\}$

$$[2;1,1] = \left\{ \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 2 \end{tikzcd} \right\} \qquad [1;2] = \left\{ \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] \arrow[r, bend right=60] \arrow[r, bend right=40, phantom, "\Downarrow"] & 1 \end{tikzcd} \right\}$$

$$[3; 1,0,3] = \left\{ \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \arrow[r] & 2 \arrow[r, bend left=60] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow"] \arrow[r, phantom, "\Downarrow"] \arrow[r, phantom, "\Downarrow"] & 3 \end{tikzcd} \right\}$$

Complete Segal Θ_2-space: Θ_2-space $W : \Theta_2^{op} \to (sSet)_{Kan}$ st

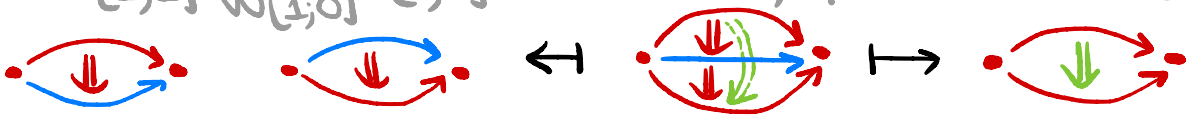
– **Segal**: (i) $W_{[n;m_1,\dots,m_n]} \xrightarrow{\sim} W_{[1;m_1]} \times_{W_{[0]}} \cdots \times_{W_{[0]}} W_{[1;m_n]}$ is a w.e. in $(sSet)_{Kan}$

(ii) $W_{[1;m]} \xrightarrow{\sim} W_{[1;1]} \times_{W_{[1;0]}} \cdots \times_{W_{[1;0]}} W_{[1;1]}$ is a w.e. in $(sSet)_{Kan}$

$W_{[0]}$ = sp. of objects; $W_{[1;0]}$ = sp. of morphisms; $W_{[1;1]}$ = sp. of 2-morphisms

Horizontal comp: $W_{[1;1]} \times_{W_{[0]}} W_{[1;1]} \xleftarrow{\sim} W_{[2;1,1]} \longrightarrow W_{[1;1]}$ (i)

Vertical comp: $W_{[1;1]} \times_{W_{[1;0]}} W_{[1;1]} \xleftarrow{\sim} W_{[1;2]} \longrightarrow W_{[1;1]}$ (ii)



– **Complete**: $W_{[0]} \xrightarrow{\sim} W^{heq}_{[1;0]}$ & $W_{[1;0]} \xrightarrow{\sim} W^{heq}_{[1;1]}$ are w.e. in $(sSet)_{Kan}$

1) $(sSet^{\Theta_2^{op}})_{\Theta_2 CSS}$ [Rezk]

2) Zig-zag of QEs btw $(sSet^{\Theta_2^{op}})_{\Theta_2 CSS}$ and $(sSet^{\Delta^{op}}\text{-}Cat)_{CSS}$ [Bergner-Rezk]

3) $N_\Theta : 2Cat \longrightarrow (sSet^{\Theta_2^{op}})_{\Theta_2 CSS}$

$(N_\Theta \mathcal{A})_{\theta,k} = Psd(\theta \times I[k], \mathcal{A})$ ← set of normal pseudo-functors

Limit in a 2-category

$F: I \to \mathcal{A}$ – 2-functor

<u>2-limit of F</u>: (L, λ) of an object $L \in \mathcal{A}$ and a cone $\lambda: \Delta L \Rightarrow F$ st

$$\lambda_*: \mathcal{A}(X, L) \xrightarrow{\cong} \mathcal{A}^I(\Delta X, F)$$

is an isomorphism of categories $\forall X \in \mathcal{A}$

<u>Slice 2-category</u>

$$\begin{tikzcd} \mathcal{A}/_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}^{[1]} \arrow[d] \\ \mathcal{A} \arrow[r, "(id_{\mathcal{A}}, \{T\})"'] & \mathcal{A} \times \mathcal{A} \end{tikzcd}$$

<u>2-category of cones</u>

$$\begin{tikzcd} \Delta \downarrow F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}^{I \times [1]} \arrow[d] \\ \mathcal{A} \arrow[r, "(\Delta, \{F\})"'] & \mathcal{A}^I \times \mathcal{A}^I \end{tikzcd}$$

<u>2-terminal object</u>: object $T \in \mathcal{A}$ satisfying one of TFE:

(i) $\mathcal{A}(X, T) \cong [0]$ is an isomorphism of categories

(ii) $\mathcal{A}/_T \xrightarrow{\cong} \mathcal{A}$ is an isomorphism of 2-categories

However: 2-limit of F $\not\Leftrightarrow$ 2-terminal object in $\Delta \downarrow F$

[Clingman–M.]

Need a "shifted" 2-category of cones to make such a characterization

$\mathrm{Ar} : \mathrm{Cat} \longrightarrow \mathrm{Cat}, \quad \mathcal{C} \longmapsto \mathcal{C}^{[1]}$

$\mathrm{Ar}_* : 2\mathrm{Cat} \longrightarrow 2\mathrm{Cat}, \quad \mathcal{A} \longmapsto \mathrm{Ar}_*\mathcal{A}$: same objects as $\mathcal{A}$ and

$(\mathrm{Ar}_*\mathcal{A})(A,B) = \mathrm{Ar}(\mathcal{A}(A,B)) = \mathcal{A}(A,B)^{[1]}$

<u>Shifted 2-category of cones</u>

$$\begin{tikzcd}
\mathrm{Ar}_*\Delta\downarrow F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathrm{Ar}_*(\mathcal{A}^I))^{[1]} \arrow[d] \\
\mathrm{Ar}_*\mathcal{A} \arrow[r, "(\mathrm{Ar}_*\Delta, \{F\})"'] & \mathrm{Ar}_*(\mathcal{A}^I) \times \mathrm{Ar}_*(\mathcal{A}^I)
\end{tikzcd}$$

objects in $\mathrm{Ar}_*\Delta\downarrow F$

$$\left(X \in \mathcal{A},\ \begin{tikzcd} \Delta X \arrow[r, bend left, shift left=2] \arrow[r, bend right, shift right=2] \arrow[r, phantom, "\Downarrow"] & F \end{tikzcd}\right)$$

<u>Theorem</u> (Clingman–M.)

A pair $(L \in \mathcal{A}, \lambda : \Delta L \Rightarrow F)$ is a 2-limit of F $\Longleftrightarrow$ (L, id_λ) is 2-terminal in $\mathrm{Ar}_*\Delta\downarrow F$.

Limit in an $(\infty,2)$-category

Enriched version

As in the case of Kan-enriched categories:

Can make sense of a

homotopy limit in a qCat- and CSS-enriched category

with universal property expressed by a

w.e. in $(\mathrm{sSet})_{\mathrm{qCat}}$ or $(\mathrm{sSet}^{\Delta^{op}})_{\mathrm{CSS}}$, respectively

And they correspond to each other via the QE:

$$(\mathrm{sSet}^{\Delta^{op}}\text{-}\mathrm{Cat})_{\mathrm{CSS}} \underset{(\iota_1^*)_*}{\overset{(\pi_1^*)_*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} (\mathrm{sSet}\text{-}\mathrm{Cat})_{\mathrm{qCat}}$$

Rem: Gagna-Harpaz-Lanari: Limits in ∞-bicategories

Terminal version

Work in progress with Nima Rasekh & Martina Rovelli.

W-complete Segal Θ_2-space

$x \in W_{[0],0}$ is a terminal object if $W_{/x} \xrightarrow{\sim} W$ is a w.e. in $(sSet^{\Theta_2^{op}})_{\Theta_2 CSS}$

Need to make sense of Ar_* in the $(\infty,2)$-setting:

$Ar_*: 2Cat \to 2Cat$ has a left adjoint $L: 2Cat \to 2Cat$

Define:

$$\begin{tikzcd}
\Theta_2 \arrow[r, hook] \arrow[d, "y"'] & 2Cat \arrow[r, "L"] & 2Cat \arrow[r, "N_\Theta"] & sSet^{\Theta_2^{op}} \arrow[dll, bend left, "Ar_*^\infty"] \\
sSet^{\Theta_2^{op}} \arrow[urr, bend left, "L^\infty"] \arrow[urrr, phantom, "\perp"] & & &
\end{tikzcd}$$

$f: I \to W \in sSet^{\Theta_2^{op}}$

Limit of f: terminal object in

$$\begin{tikzcd}
Ar_*^\infty \Delta \downarrow f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (Ar_*^\infty(W^I))^{N_\Theta[I;0]} \arrow[d] \\
Ar_*^\infty W \arrow[r, "(Ar_*^\infty \Delta, \{f\})"'] & Ar_*^\infty(W^I) \times Ar_*^\infty(W^I)
\end{tikzcd}$$

We aim to show that:

- this gives a way of defining limits in $(\infty,2)$-categories in a model-independent way
- it corresponds to the homotopy limits defined in enriched models
- add a weight $\rightsquigarrow$ lax limits

Thank you for your attention!