The mathematics of conformal field theory

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General problem: Quantum field theory is one of the most important physical discoveries of the last century. But the mathematical foundations are far from solid.

Goal for this talk: Focus on the case of conformal field theory; discuss three mathematical approaches, their strengths, and their weaknesses.

Plan for the talk:

1. Background/motivation: Mathematical approaches to field theory
2. Overview of three mathematical approaches to conformal field theory
3. Comparison and connections between the three approaches

Section 1

Mathematical approaches to field theory

Warning: I am not a physicist, and this is not a physics talk. So we will be a little bit imprecise before we get to the mathematics.

Mathematical approach to classical field theory

- Manifold X (space-time).
- Vector bundle $E \to X$; vector space of global sections $\Gamma(X, E)$ (space of fields).
- System of PDEs on $\Gamma(X, E)$ (equations of motion) cutting out the permissible configurations in the physical system, denoted $\mathsf{Sol}(X)$.
- Taking a measurement of the system corresponds to a function $\mathsf{Sol}(X) \to \mathbb{C}$ (observable).

Mathematical approach to quantum field theory

Think of a quantum state as a vector in a Hilbert space H.

A quantum observable is a linear operator $A : H \to H$.

The "value" of such an observable is defined only on states which are **eigenvectors** for the operator:

$$A|\psi\rangle = \lambda|\psi\rangle.$$

In this case the value of the observable is the eigenvalue λ.

Uncertainty Principle: We can measure momentum or position, but not both.

$\longleftrightarrow$ The momentum and position operators do not have simultaneous eigenvectors.

Mathematical approach to quantum field theory [Costello–Gwilliam]

Let $\mathcal{T}$ be a quantum field theory on space-time X. For $U \subset X$, let

$$\mathcal{F}(U) := \text{“observables of } \mathcal{T} \text{ on } U\text{”}$$
$$= \text{operators depending only on the behaviour of a field over space-time interval } U.$$

- Then it follows that for $U \subset V$, we have $\mathcal{F}(U) \hookrightarrow \mathcal{F}(V)$.
- In general, taking measurements disturbs the system. So taking a measurement on U_1 and then another measurement on U_2 is not the same as taking a measurement on $U_1 \cup U_2$. But it's okay if U_1 and U_2 are disjoint (in space-time).

$$\mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \to \mathcal{F}(U_1 \cup U_2).$$

We say that $\mathcal{F}$ is a factorizable cosheaf on X [CG, Lurie].

The theory of factorizable cosheaves gives a rigorous mathematical model for the study of quantum field theory.

Problem: For many quantum field theories $\mathcal{T}$, we just don't know how to write down the quantum observables $\mathcal{F}(U)$.

From now on, we'll restrict our attention to conformal field theories, where the structures are invariant under conformal transformations of the space-time manifold X.

- This additional symmetry gives us more mathematical tools to work with.

Section 2

Overview of the three approaches

	Factorizable cosheaves	**Vertex algebras**	**Chiral/factorization algebras**
Mathematical flavour	* Topology * Differential geometry	* Analysis	* Algebraic geometry
	Cosheaf (on a base X) + extra structure	Vector space + extra structure	Sheaf (on a base X) + extra structure
	$U \mapsto \mathcal{F}(U)$		Lie algebra/coalgebra
Physical content	Quantum observables of the field theory	Symmetries of a 2d conformal field theory	Collisions between local operators

Vertex algebras

Roughly, a vertex algebra is an algebra V equipped with meromorphic multiplication, parametrized by the complex plane:

$$V \otimes V \to V((z)).$$

More precisely, we have a map (the vertex operators)

$$\begin{aligned} Y(\cdot, z) : V &\to \operatorname{End} V[\![z, z^{-1}]\!] \\ A &\mapsto Y(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}. \end{aligned}$$

Example: a commutative vertex algebra

Take V a commutative algebra equipped with a derivation $T : V \to V$.

Then V has a structure of vertex algebra, where

$$
\begin{aligned}
Y(A, z) &= e^{zT} A \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} T^k A \cdot z^k.
\end{aligned}
$$

In other words, for $n < 0$, $A_{(n)}$ is given by multiplication in V by

$$\frac{1}{(-n-1)!} T^{-n-1} A.$$

Vertex algebras in mathematics

Inspired by work of I. Frenkel, R. Borcherds noticed that for any lattice, one can construct a space V acted on by operators corresponding to lattice vectors.

$$V = \mathbb{C}[L] \otimes \mathrm{Sym}(L(1) \oplus L(2) \oplus L(3) \cdots)$$

In fact, there are operators ('vertex operators') for each element of V.

- This is a *lattice vertex algebra*.
- Borcherds formalized the properties satisfied by these operators to come up with the definition of a vertex algebra.

Vertex algebras explain surprising phenomena in mathematics

Example 1: Monstrous Moonshine

The Monster $\mathbb{M}$ is the largest sporadic simple group.

It was predicted to exist and we even knew some of its properties before we knew how to construct it! [Griess 1982]

[Frenkel–Lepowsky–Meurman 1988]: $\mathbb{M}$ is the automorphism group of a vertex algebra constructed from the Leech lattice.

> So the question "What is the monster" now has several reasonable answers:
>
> 1 The monster is the largest sporadic simple group, or alternatively the unique simple group with its order.
>
> 2 It is the automorphism group of the Griess algebra.
>
> 3 It is the automorphism group of the monster vertex algebra. (This is probably the best answer.)
>
> 4 It is a group of diagram automorphisms of the monster Lie algebra.

— Borcherds, "What is the Monster" 2002

Conway and Norton's Moonshine Conjecture [1979]:
The representation theory of the Monster group $\mathbb{M}$ is related to modular functions.

- Proved by Borcherds in 1992 (Fields medal in 1998), using the Monster vertex algebra as a bridge between $\mathbb{M}$ and modular functions.

Example 2: Modular tensor categories

Important fact in representation theory:
Certain categories of representations of affine Lie algebras and quantum groups form modular tensor categories.
[Kazhdan–Lusztig 1993, Finkelberg 1996]

This turns out to be a special case of the following general result:

- If V is a sufficiently nice vertex algebra ('rational') its representation category is a modular tensor category. [Huang 2005]

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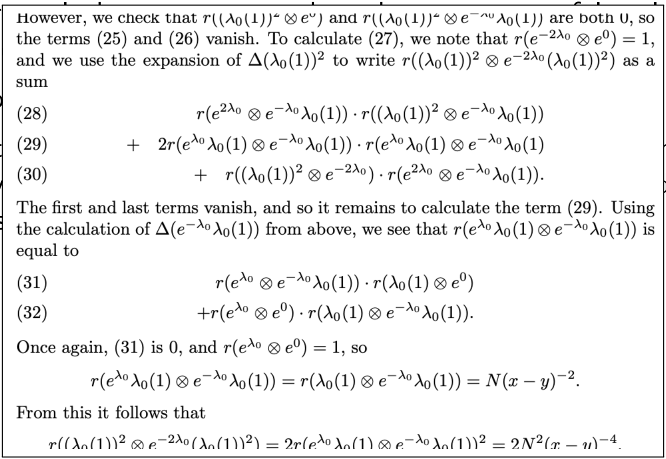


However, we check that $r((\lambda_0(1))^2 \otimes e^0)$ and $r((\lambda_0(1))^2 \otimes e^{-\lambda_0}\lambda_0(1))$ are both 0, so the terms (25) and (26) vanish. To calculate (27), we note that $r(e^{-2\lambda_0} \otimes e^0) = 1$, and we use the expansion of $\Delta(\lambda_0(1))^2$ to write $r((\lambda_0(1))^2 \otimes e^{-2\lambda_0}(\lambda_0(1))^2)$ as a sum

$$r(e^{2\lambda_0} \otimes e^{-\lambda_0}\lambda_0(1)) \cdot r((\lambda_0(1))^2 \otimes e^{-\lambda_0}\lambda_0(1)) \tag{28}$$

$$+ \quad 2r(e^{\lambda_0}\lambda_0(1) \otimes e^{-\lambda_0}\lambda_0(1)) \cdot r(e^{\lambda_0}\lambda_0(1) \otimes e^{-\lambda_0}\lambda_0(1) \tag{29}$$

$$+ \quad r((\lambda_0(1))^2 \otimes e^{-2\lambda_0}) \cdot r(e^{2\lambda_0} \otimes e^{-\lambda_0}\lambda_0(1)). \tag{30}$$

The first and last terms vanish, and so it remains to calculate the term (29). Using the calculation of $\Delta(e^{-\lambda_0}\lambda_0(1))$ from above, we see that $r(e^{\lambda_0}\lambda_0(1) \otimes e^{-\lambda_0}\lambda_0(1))$ is equal to

$$r(e^{\lambda_0} \otimes e^{-\lambda_0}\lambda_0(1)) \cdot r(\lambda_0(1) \otimes e^0) \tag{31}$$

$$+r(e^{\lambda_0} \otimes e^0) \cdot r(\lambda_0(1) \otimes e^{-\lambda_0}\lambda_0(1)). \tag{32}$$

Once again, (31) is 0, and $r(e^{\lambda_0} \otimes e^0) = 1$, so

$$r(e^{\lambda_0}\lambda_0(1) \otimes e^{-\lambda_0}\lambda_0(1)) = r(\lambda_0(1) \otimes e^{-\lambda_0}\lambda_0(1)) = N(x-y)^{-2}.$$

From this it follows that

$$r((\lambda_0(1))^2 \otimes e^{-2\lambda_0}(\lambda_0(1))^2) = 2r(e^{\lambda_0}\lambda_0(1) \otimes e^{-\lambda_0}\lambda_0(1))^2 = 2N^2(x-y)^{-4}.$$

— C. "Chiral algebras, factorization algebras, and Borcherds's "singular commutative rings approach to vertex algebras"" 2019.

Geometers tell us that this is often not the best way to get intuition for the global picture . . .

Factorization algebras/chiral algebras

[Beilinson–Drinfeld 1990s]: introduced chiral/factorization algebras, a coordinate-free reformulation and generalization of the notion of a vertex algebra.

[Francis–Gaitsgory 2011]: generalized the definitions and basic results to work in arbitrary complex dimension.

[C. 2015]: first non-trivial higher dimensional examples in the literature.

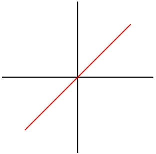
Motivation

Let X be a smooth complex variety.

Recall that in conformal field theory, we're interested in local operators living at a collection of points

$$(x_1, \dots x_n) \in X^n,$$

and we want to understand what happens when these points collide—that is, when we approach the diagonal $\Delta \subset X^n$.



Definition: A factorization space over X

1. For every $n \in \mathbb{N}$ a space $\mathcal{Y}_n \to X^n$.
2. **Ran's condition.** For example, consider the inclusion $\Delta : X \hookrightarrow X^2$. We require

$$\nu : \mathcal{Y}_1 \xrightarrow{\sim} \mathcal{Y}_2|_X .$$

3. **Factorization isomorphisms.** For example, let $j : \Delta^c \hookrightarrow X^2$ be the complement of the diagonal. We require

$$c : \mathcal{Y}_2|_{\Delta^c} \xrightarrow{\sim} (\mathcal{Y}_1 \times \mathcal{Y}_1)\,|_{\Delta^c} .$$

Remark: This is an infinite-dimensional phenomenon.

Factorization algebras and chiral algebras

A factorization algebra is a linear analogue of a factorization space.

- In particular, we have sheaves on each X^n instead of spaces.
- We can produce factorization algebras by starting with a factorization space and then *linearizing* (e.g. taking cohomology).

A chiral algebra is an equivalent, Koszul-dual reformulation of the axioms, consisting of a single sheaf on X equipped with some extra structure.

Example 0: a discrete example to warm up

Define $\mathcal{Y}_n$ to consist of tuples $(\mathbf{x}, (m_x)_{x \in \{\mathbf{x}\}})$, where

- $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_i \in X$;
- to each distinct point x in the set $\{x_1, \ldots, x_n\}$ we assign a label $m_x \in \mathbb{Z}$;
- the map $\mathcal{Y}_n \to X^n$ is the forgetful map $(\mathbf{x}, (m_x)_{x \in \{\mathbf{x}\}}) \mapsto \mathbf{x}$.

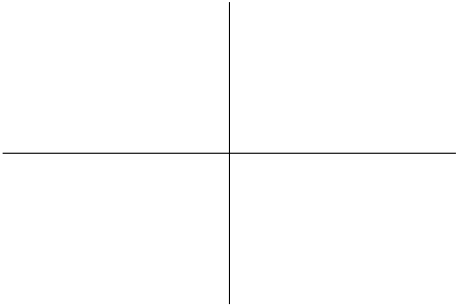
Idea: When two points collide, we add the value of the labels.

Example 1: the Hilbert scheme of points

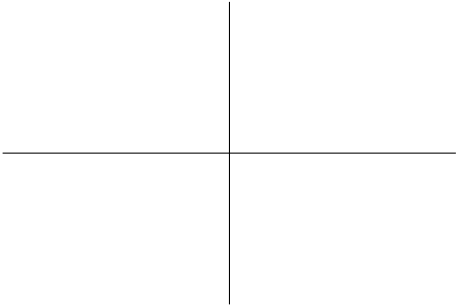
Hilb_X parametrizes 0-dimensional subschemes of X of finite length.

For example, for $X = \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, y]$, here is a subscheme of length 2:

$$\operatorname{Spec} \mathbb{C}[x, y]/(x, y(y-1))$$

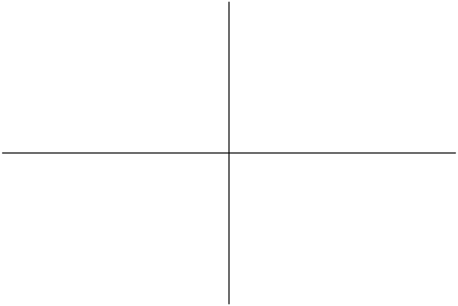


$$\operatorname{Spec} \mathbb{C}[x, y]/(x, y(y - \lambda))$$

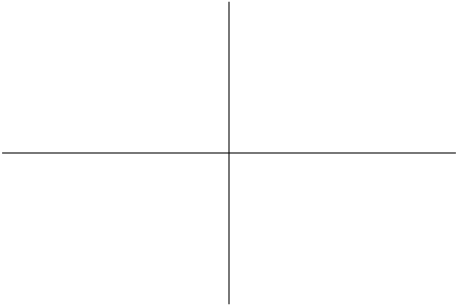


(as long as $\lambda \neq 0$)

$\operatorname{Spec} \mathbb{C}[x, y]/(x, y^2)$



$$\operatorname{Spec} \mathbb{C}[x, y]/(x^2, y)$$



The Hilbert scheme factorization space

Define $\mathcal{H}ilb_{X^n}$ to be the space parametrizing pairs $(\mathbf{x}, \xi)$, where

- $\mathbf{x} = (x_1, \dots x_n) \in X^n$
- $\xi \in \text{Hilb}_X$ is supported set-theoretically on the set $\{x_1, \dots x_n\}$.

Theorem (C.)

This is a factorization space over X.

Example 2: the Beilinson–Drinfeld Grassmannian

Associated to a smooth curve X and a reductive group G, we define a factorization space built out of principal G-bundles on X.

Gr_{G,X^n} is the space parametrizing triples $(\mathbf{x}, \mathcal{P}, \sigma)$, where

- $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$;
- $\mathcal{P} \to X$ is a principal G-bundle on X;
- σ is a trivialization of $\mathcal{P}$ on $X \setminus \{x_1, \ldots, x_n\}$.

[Beilinson–Drinfeld 1991, Gaitsgory 2012] This is a big deal in the geometric Langlands program.

Fibres of the Beilinson–Drinfeld Grassmannian

Consider the fibre of $Gr_{G,X}$ over a point $x \in X$:

$$\{(x, \mathcal{P}, \sigma\}$$

$\mathcal{P}$ is trivial on $X \setminus \{x\}$ and the formal disc $\widehat{D}_x \cong \operatorname{Spec} C[\![t]\!]$,

so the data of $\mathcal{P}, \sigma$ is determined by a gluing isomorphism over the punctured formal disc $D_x^\circ \cong \operatorname{Spec} C((t))$ pause

$$\phi : \operatorname{Spec} C((t)) \to G.$$

Note that if ϕ extends to $\widehat{D}_x$, $\mathcal{P}$ is trivial.

$$Gr_{G,X,x} \cong G((t))/G[\![t]\!]$$ the affine Grassmannian

Application: representations of factorization spaces

Linearizing with respect to line bundles on Gr_{G,X^n} gives rise to factorization algebras associated to affine Lie algebras.

- One hopes to recover the representations of these affine Lie algebras (in particular, *integrable representations* of a fixed level) geometrically. (Ongoing joint work with Kobi Kremnitzer.)
- We introduce the notion of a module over a factorization space; linearizing gives rise to modules of the corresponding factorization algebras.
- We construct examples of modules over the Beilinson–Drinfeld Grassmannian using moduli spaces of parabolic G-bundles; this allows us to give geometric constructions of representations of affine Lie algebras and their tensor products.

Section 3

Comparisons and connections

Advantages and disadvantages

	Factorizable cosheaves $U \mapsto \mathcal{F}(U)$	**Vertex algebras** $(V, Y(\cdot, z))$	**Chiral/factorization algebras** $\{\mathcal{A}_n \in \mathcal{D}(X^n)\}_n$
Dimension	Any real dimension	Limited to 2d	Any complex dimension
Examples	Many examples are unknown	Many known examples with successful applications	Examples can be constructed from geometric spaces
Calculations	Natural and predicted by physics	Explicit and concrete; involved and unmotivated	Elegant and intuitive; inexplicit

Key principle: In order to solve a broad range of problems in the mathematics of conformal field theory, we should work to combine all three approaches.

Vertex algebras and factorization algebras

Roughly: vertex algebras are factorization algebras over curves.

More precisely, let $(V, Y(\cdot, z))$ be a vertex algebra.

It is quasi-conformal if it has a nice action of the group
$G = \mathrm{Aut}(\mathrm{Spf}\,\mathbb{C}[\![t]\!])$.

In physics, this corresponds to Virasoro symmetry of the CFT.

Given any smooth curve C, the quasi-conformal structure allows us to "spread" the vector space over the curve to obtain a sheaf $\mathcal{V}_C$.

Theorem (Frenkel–Ben-Zvi)

$\mathcal{V}_C$ has the structure of a chiral algebra over the curve C.

Moreover, the assignment $C \mapsto \mathcal{V}_C$ is compatible with pullback along étale morphisms between smooth curves.

We say that the assignment $C \mapsto \mathcal{V}_C$ is a universal chiral algebra of dimension one.

Theorem (FBZ, see also Huang–Lepowsky)

This construction yields an equivalence of categories

$$\left\{ \begin{array}{c} \textit{quasi-conformal} \\ \textit{vertex algebras} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{universal chiral algebras} \\ \textit{of dim. 1} \end{array} \right\}.$$

		Chiral/factorization algebras		
Vertex algebras $(V, Y(\cdot, z))$		**Chiral algebras** $(\mathcal{A}_1, \mu)$	[BD, FG] ⟷	**Factorization algebras** $\{\mathcal{A}_n \in \mathcal{D}(X^n)\}$
Quasi-conformal vertex algebras	[FBZ, HL] ⟷	Universal chiral algebras of dimension 1	[C] ⟷	Universal factorization algebras of dimension 1 ??
??		Universal chiral algebras of dimension d	[C] ⟷	Universal factorization algebras of dimension d ??
Representations of the group G_d of automorphisms of the d-dimensional formal disk	[BD, C] ⟷	Universal $\mathcal{D}$-modules of dimension d		

Factorizable cosheaves and vertex algebras

Theorem (Costello–Gwilliam)

There is a functor from the category of certain nice factorizable cosheaves on $\mathbb{A}^1$ to the category of vertex algebras.

Corollary

These nice factorizable cosheaves on $\mathbb{A}^1$ give rise to factorization algebras on $\mathbb{A}^1$.

Work in progress [C.–Gwilliam]: For suitable adjectives, holomorphic factorizable cosheaves over X are the same as factorization algebras over X, for X a smooth complex variety of any dimension.

Factorizable cosheaves ⟷ Factorization algebras

Thank you!

Vertex algebras