FINITENESS PROPERTIES OF MODULI SPACES OF HIGH-DIMENSIONAL MANIFOLDS

Abstract. These are Mauricio Bustamante's notes for his talk at the **"Viva Talbot!: A virtual MIT Talbot Workshop retrospective"** in 2021.

1. Moduli spaces

Fix a smooth manifold M. The following task is typically given: classify smooth M-bundles over a CW-complex B up to isomorphism.

Example. (A success story). Suppose $M = \mathbb{R}^d$. Then there is a bijection [exercise]

$$\{\text{smooth } \mathbb{R}^d\text{-bundles}\}/\text{iso} \xleftarrow{\cong} \{\text{rank } d\text{-vector bundles over } B\}/\text{iso}$$

The classification problem is now solved roughly as follows: we construct a "classifying space" $\mathcal{M}_d$ and a bijection of sets

$$[B, \mathcal{M}_d] \to \{\text{rank } d\text{-vector bundles over } B\}/\text{iso}$$

This classifying space is nothing but a Grassmann manifold. In fact, we define

$$\mathcal{M}_{d,k} \simeq \{V \subset \mathbb{R}^{d+k} \mid V \text{ is a vector subspace isomorphic to } \mathbb{R}^d\}.$$

This space is slightly annoying to topologize, so we can think of it instead as

$$\mathcal{M}_{d,k} \simeq \mathrm{Inj}(\mathbb{R}^d, \mathbb{R}^{d+k})/\mathrm{GL}_d(\mathbb{R})$$

where $\mathrm{Inj}(\mathbb{R}^d, \mathbb{R}^{d+k})$ is the space of injective linear maps from $\mathbb{R}^d$ to $\mathbb{R}^{d+k}$ (a.k.a the Stiefel manifold) and the action of $\mathrm{GL}_d(\mathbb{R})$ on it is by precomposition. Note that $\mathrm{Inj}(\mathbb{R}^d, \mathbb{R}^{d+k})$ can be seen as a subspace of $(\mathbb{R}^{d+k})^d$, and the orbit space is naturally given the the quotient topology.

It is customary to write

$$\mathcal{M}_d := \operatorname*{colim}_k \mathcal{M}_{d,k} =: B\mathrm{GL}_d(\mathbb{R}).$$

The general case is done similarly: we create a "Grassmann-type" moduli space, namely we define it as the space of *embedded* submanifolds of $\mathbb{R}^{d+k}$

$$\mathcal{M}_{M,k} \simeq \{W \subset \mathbb{R}^{d+k} \mid W \text{ is a smooth submanifold diffeomorphic to } M\}$$

Good luck topologizing this thing (one can, though). So here is an alternative

$$\mathcal{M}_{M,k} \simeq \mathrm{Emb}(M, \mathbb{R}^{d+k})/\operatorname{Diff}(M)$$

where $\mathrm{Emb}(M, \mathbb{R}^{d+k})$ is the space of smooth embeddings of M into $\mathbb{R}^{d+k}$ with the C^∞-topology and $\operatorname{Diff}(M)$ is the (topological) group of diffeomorphisms of M.

The moduli space of d-dimensional closed smooth manfiolds diffeomorphic to M is

$$B\operatorname{Diff}(M) := \operatorname*{colim}_k \mathcal{M}_{M,k}$$

There is indeed a correspondence

$$[B, B\operatorname{Diff}(M)] \to \{\text{smooth } M\text{-bundles over } B\}/\text{iso}$$

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1.1. **(Co)homology and homotopy groups of moduli spaces.** This correspondence with sets of homotopy classes of maps to $B\,\mathrm{Diff}(M)$ is useful if we know something about the homotopy type of the latter. Now, we don't expect to have uniform answer for *all* manifolds M. So we lower our expectations and ask, what can we say in general about the (co)homology and homotopy groups of $B\,\mathrm{Diff}(M)$?

These are the most important invariants for geometric purposes:

(1) $H^*(B\,\mathrm{Diff}(M))$ is the ring of characteristic classes of smooth M-bundles (by the Yoneda lemma).
(2) $\pi_*(B\,\mathrm{Diff}(M))$ is the set (actually group) of isomorphism classes of smooth M-bundles over spheres.

Knowing these invariants is important not only for the classification problem posed above, but also for discovering new manifold topology phenomena and defining new invariants. For example:

- $H^*(B\mathrm{GL}_d(\mathbb{R})) \cong \mathbb{Z}\oplus\cdots\oplus\mathbb{Z}\oplus\mathbb{Z}/2\oplus\cdots\oplus\mathbb{Z}/2$. Thanks to this calculation we can derive a bunch of things about bordism groups and prove the signature theorem.
- $\pi_4(B\mathrm{GL}_4(\mathbb{R})) \cong \mathbb{Z}\oplus\mathbb{Z}$. This calculation was the first step in Milnor's discovery of exotic 7-spheres.

But again, we don't really expect to give "a calculation" of the cohomology and homotopy groups of $B\,\mathrm{Diff}(M)$ for all manifolds. Nevertheless, here is something that holds for a very large class of manifolds and that could serve as an input to make computations for a specific M.

Theorem 1.1 (B–Krannich–Kupers)**.** *Let M be a closed connected oriented smooth manifold of dimension $d = 2n \geq 6$. If $\pi_1(M)$ is finite, then for all $k \geq 2$ the groups $\pi_k(B\,\mathrm{Diff}^+(M))$ and $H^{k-1}(B\,\mathrm{Diff}^+(M);\mathbb{Z})$ are finitely generated.*

Remark. The case $k = 1$, that is $\pi_1(B\,\mathrm{Diff}^+(M)) = \pi_0(\mathrm{Diff}^+(M)) = \{$ diffeomorphisms of $M\}$/isotopy, was done by Sullivan (for $\pi_1(M) = 0$) and Triantafillou (for $\pi_1(M)$ finite). In fact they show that the mapping class group is a group commensurable with an arithmetic group. In particular it is of type F_∞.

In this talk we will discuss the strategy that leads to the proof of this theorem. But first, we will discuss briefly what's the situation in low dimensions, and how people could have approached this problem in the 70's and 80's.

2. What do we know in low dimensions?

The conclusion of Theorem 1.1 holds true in dimensions 2 and 3. Let us only discuss finite generation of the homotopy groups of $B\,\mathrm{Diff}(M)$.

Dimension 2. The only case to look at is S^2. In this case Smale proved that the inclusion $SO(3) \hookrightarrow \mathrm{Diff}(S^2)$ is a homotopy equivalence, and so must be the induced map $BSO(3) \to B\,\mathrm{Diff}(S^2)$.

Dimension 3. The case of S^3 is similar to that of S^2. For Hatcher showed that the inclusion $SO(4) \hookrightarrow \mathrm{Diff}(S^3)$ is a homotopy equivalence. By Perelman's work, the remaining cases are orbit spaces of isometric finite group actions on the round 3-sphere. For these spherical 3-manifolds, Bamler and Kleiner showed that the analog of Hatcher and Sullivan's result for spheres is true in more generality: for any Riemannian metric g on S^3/G of constant sectional curvature $= 1$, they show that the inclusion

$$\mathrm{Isom}(S^3/G, g) =: \mathrm{Isom}(S^3/G) \hookrightarrow \mathrm{Diff}(S^3/G)$$

is a homotopy equivalence.

Therefore

$$B\mathrm{Isom}(S^3/G) \simeq B\,\mathrm{Diff}(S^3/G).$$

So we have that

$$\pi_k(B\,\mathrm{Diff}(S^3/G)) \cong \pi_k(B\mathrm{Isom}(S^3/G)) \cong \pi_{k-1}(\mathrm{Isom}(S^3/G))$$

Now observe that since $\mathrm{Isom}(S^3/G)$ is a compact Lie group, then $\pi_0(\mathrm{Isom}(S^3/G))$ is a finite group and $\pi_1(\mathrm{Isom}(S^3/G))$ is a finitely presented group. For the higher homotopy groups, we look at the identity component $\mathrm{Isom}_0(S^3/G)$. This is a simply connected compact Lie group and hence its homology groups are finitely generated. Therefore its homotopy groups are finitely generated by a result of Serre.

Dimension 4. The theorem fails in dimension 4. Baraglia showed that for a $K3$-surface K (which is simply-connected) the group $\pi_2(B\,\mathrm{Diff}(K))$ has infinite rank.

3. High-dimensions in the 70-80's

Waldhausen showed[1] that the algebraic K-theory (let's say space) of M splits as

$$A(M) = K(\mathbb{S}[\Omega M]) \simeq \Omega^\infty \Sigma^\infty M_+ \times \mathrm{Wh}(M)$$

and furthermore that $\Omega_0 \mathrm{Wh}(M) \simeq B\mathscr{P}(M)$, where $B\mathscr{P}(M)$ is the classifying space of stable pseudoisotopies of M, which is defined as the colimit of the maps

$$\cdots \to BP(M \times D^k) \to BP(M \times D^{k+1}) \to \cdots$$

and $P(M)$ is the topological group of diffeomorphisms of $M \times I$ which restrict to the identity on $M \times \{0\} \cup \partial M \times I$.

On the other hand, Igusa showed that the map

$$BP(M) \to B\mathscr{P}(M)$$

is about $d/3$-connected[2]. The passage from $BP(M)$ to $B\,\mathrm{Diff}(M)$ is essentially through surgery theory, the understanding of the space of homotopy automorphisms of M and something known as the Hatcher spectral sequence.

All these ingredients can be combined with a theorem of Betley which asserts that the homotopy groups of $A(M)$ are finitely generated if $\pi_1(M)$ is finite, to conclude that the homotopy groups of $B\,\mathrm{Diff}(M)$ are finitely generated in degrees up to roughly $d/3$.

That's as much as we can say by this method.

4. High-dimensions after Galatius–Randal-Williams and Goodwillie–Klein–Weiss

To improve on what people did in the 80's, we will use three results:

(1) **Galatius–Randal-Williams and Friedrich [GRW-F].** They show that if M is a compact smooth manifold of dimension $d = 2n \geq 6$ with finite fundamental group, then
   - $H_k(B\,\mathrm{Diff}_\partial(M\# W_g))$, where $W_g = \#^g S^n \times S^n$, becomes independent of g for g large, i.e. these groups exhibit homological stability.
   - The stable homology is isomorphic to the homology of a homotopy quotient of a component of the infinite loop space $\Omega_0^\infty \mathrm{MT}\theta$ of a certain Thom spectrum.

   Putting together these two facts one can show that, under some additional conditions on the connectivity of the inclusion $\partial(M\# W_g) \hookrightarrow M\# W_g$, the groups $H_k(B\,\mathrm{Diff}_\partial(M\# W_g))$ are finitely generated if $\pi_1(M)$ is finite, provided $g \gg k$.

(2) **Goodwillie–Klein–Weiss [GKW].** They show that if M^d and N^d are compact manifolds with boundary, and if M can be built from a disk by attaching handles of index $\leq d-3$ then

   $$\mathrm{Emb}(M, N) \simeq \lim(T_1 \leftarrow T_2 \leftarrow \cdots)$$

   where $T_1 = \mathrm{Bun}(TM, TN)$ is the space of bundle isomorphisms between the tangent bundles of M and N, and the maps $T_k \to T_{k-1}$ are fibrations whose fibers L_k are certain section spaces of a fiber bundle whose base space and fiber are made out of configuration spaces of k points in M and N. The point is that layers L_k have an explicit enough description to conclude, by an inductive argument, that if $\pi_1(M)$ and $\pi_1(N)$ are both finite, then the homotopy groups $\pi_k(\mathrm{Emb}(M, N))$ are finitely generated.

[1] Actually a complete proof appeared in 2012 by Waldhausen–Jahren–Rognes.

[2] Conjecturally this is not optimal. Manuel Krannich has shown that rationally this map is about d connected for all 2-connected compact manifolds.

(3) **The Kupers–Weiss fiber sequence.**

Throughout we fix a d-dimensional compact smooth manifold M and a codimension 0 submanifold $N \subset \partial M$ of its boundary, and we thicken it to an $N \times I \subset M$, where $I = [0, 1]$.

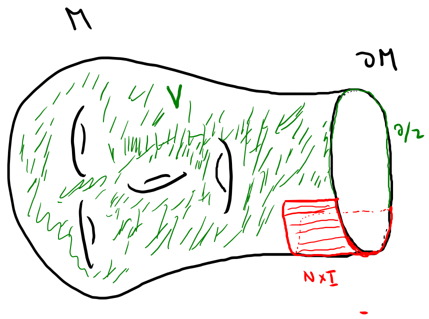
The Weiss fiber sequence expresses $B\,\mathrm{Diff}_\partial(N \times I)$ as the difference between $B\,\mathrm{Diff}_\partial(M)$ and $B\,\mathrm{Emb}^{\cong}_{\partial/2}(M)$, where $\mathrm{Emb}^{\cong}_{\partial/2}(M)$ is the topological monoid of self-embeddings of M which are the identity on a neighborhood of $\partial M - \mathrm{int}(N)$, and are isotopic (through such embeddings) to a diffeomorphism that is the identity on a neighborhood of ∂M. More precisely, the Weiss fiber sequence is

$$B\,\mathrm{Diff}_\partial(N \times I) \to B\,\mathrm{Diff}_\partial(M) \to B\,\mathrm{Emb}^{\cong}_{\partial/2}(M). \tag{1}$$

In order to obtain (1) we set $V := M - \mathrm{int}(N \times [0, 1])$. The boundary of V decomposes as

$$\partial V = (\partial M - \mathrm{int}(N)) \cup \partial_1 V$$

for some other manifold $\partial_1 V$.



We will use the notation $\partial/2 = \partial M - \mathrm{int}(N)$. By the isotopy extension theorem[3], restriction gives rise to a fibration

$$\mathrm{Diff}_\partial(M - \mathrm{int}(V)) \to \mathrm{Diff}_\partial(M) \to \mathrm{Emb}^{\mathrm{ext}}_{\partial/2}(V, M)$$

where the last term is the space of embeddings of V into M which restrict to the inclusion on $\partial/2$ and are isotopic to an embedding that extends to a self-diffeomorphism of M. Observe that the fiber of this fibration is exactly $\mathrm{Diff}_\partial(N \times I)$. The base of the fibration can be identified with $\mathrm{Emb}^{\cong}_{\partial/2}(M)$ because V and M are isotopy equivalent. In total, we get a fibration sequence

$$\mathrm{Diff}_\partial(N \times I) \to \mathrm{Diff}_\partial(M) \to \mathrm{Emb}^{\cong}_{\partial/2}(M)$$

which deloops to (1) as all the maps are compatible with the operation of composition.

It turns out that one can use the operation "stacking in the interval direction" to give a unital topological monoid model for $B\,\mathrm{Diff}_\partial(N \times I)$, and a right $B\,\mathrm{Diff}_\partial(N \times I)$-module model for $B\,\mathrm{Diff}_\partial(M)$. This gives rise to a delooping of $B\,\mathrm{Diff}_\partial(N \times I)$ and yields a *delooped Weiss fiber sequence* (established by A. Kupers)

$$B\,\mathrm{Diff}_\partial(M) \to B\,\mathrm{Emb}^{\cong}_{\partial/2}(M) \to B^2\,\mathrm{Diff}_\partial(N \times I) \tag{2}$$

This sequence is the result of a more general fact: if X is a simplicial or topological right A-module, for A some path-connected unital topological monoid, then there is a fibration sequence of the form

$$X \to X /\!\!/ A \to BA.$$

[3] It is perhaps safer to apply the isotopy extension theorem to proper embeddings. So one should change V by another isotopy equivalent manifold V' such that $\partial V' = \partial/2$. For example $V' = M - \mathrm{int}(N) \times I$.

In our case, $X = B\operatorname{Diff}_\partial(M)$ and $A = B\operatorname{Diff}_\partial(N \times I)$. The most involved step is to identify $B\operatorname{Diff}_\partial(M) /\!\!/ B\operatorname{Diff}_\partial(N \times I)$ with $B\operatorname{Emb}^{\cong}_{\partial/2}(M)$. This can be done by showing that there is a weak homotopy equivalence

$$B\operatorname{Diff}_\partial(M) /\!\!/ B\operatorname{Diff}_\partial(N \times I) \to B\operatorname{Emb}^{\cong}_{\partial/2}(M) \times * /\!\!/ B\operatorname{Diff}_\partial(N \times I) \to B\operatorname{Emb}^{\cong}_{\partial/2}(M).$$

In the end this also follows from the isotopy extension theorem.

5. The strategy to prove Theorem 1.1

We will only discuss here finite generation of the higher homotopy groups.

Firstly we pick a handle decomposition of M and write $M^{\leq 2}$ for the codimension 0 submanifold of M consisting of handles of index ≤ 2. Let K denote the closure of the complement of $M^{\leq 2}$. By restricting a diffeomorphism of M to $M^{\leq 2}$ we obtain a fibration[4]

$$\operatorname{Emb}(M^{\leq 2}, M) \to B\operatorname{Diff}_\partial(K) \to B\operatorname{Diff}(M)$$

Observe that from [GKW] the homotopy groups of the fiber are finitely generated. Thus, from the long exact sequence in homotopy groups, we see that it suffices to show that the homotopy groups of $B\operatorname{Diff}_\partial(K)$ are finitely generated.

To analyze the latter we use a Weiss fiber sequence

$$B\operatorname{Diff}_\partial(\partial K \times I) \to B\operatorname{Diff}_\partial(K) \to B\operatorname{Emb}^{\cong}(K)$$

We can apply [GKW] to the base space to conclude that its homotopy groups are finitely generated. So it suffices to show that $B\operatorname{Diff}_\partial(\partial K \times I)$ has finitely generated homotopy groups. Notice that it's just ok to show that all the homotopy groups of $B^2\operatorname{Diff}(\partial K \times I)$ are finitely generated. But this is a simply connected space and by a theorem of Serre its homotopy groups are finitely generated if and only if its homology groups are so. We will try to show this instead. The first thing to notice is that the boundary of K does not change if we "stabilize it by W_g's". So we can use the following Weiss fiber sequence

$$B\operatorname{Diff}_\partial(K\#W_g) \to B\operatorname{Emb}^{\cong}(K\#W_g) \to B^2\operatorname{Diff}_\partial(\partial K \times I)$$

Now [GRW-F] should imply that the homology groups of the fiber are finitely generated as long as g is very large (which is ok to assume here). But it's more subtle than this. Since there is a nonempty boundary which is not necessarily highly connected, [GRW-F] identifies the stable homology with a *homotopy quotient* of a component of the infinite loop space of the MT spectrum. This actually creates a major difficulty with this argument and it took us several pages to go around it. But I will omit the details here. We will just carry on assuming that the desired groups are finitely generated, but at least you know that I am lying to you at this point.

Having said that, a straightforward spectral sequence argument now tells us that the homology groups of $B^2\operatorname{Diff}_\partial(\partial K \times I)$ are finitely generated if those of $B\operatorname{Emb}^{\cong}(K\#W_g)$ are.

What we know about $B\operatorname{Emb}^{\cong}(K\#W_g)$ is that its homotopy groups are finitely generated, by [GKW]. How do we conclude anything about its homology groups? Here is a general fact:

Fact. Let X be a path-connected space such that $\pi_1(X)$ is a group of type F_∞ and its higher homotopy groups are finitely generated. Then the homology groups of X are finitely generated. □

Because of this, we are left to show that $\pi_1(B\operatorname{Emb}^{\cong}(K\#W_g)) = \pi_0(\operatorname{Diff}_\partial(K\#W_g))$ is a group of type F_∞. To do this one can extract, from the bottom of the long exact sequence of the Weiss fiber sequence, an extension of groups

$$1 \to A \to \pi_0(\operatorname{Diff}_\partial(K\#W_g)) \to \pi_0(\operatorname{Emb}^{\cong}(K\#W_g)) \to 1$$

where A is an abelian group.

[4] I should pick some components of the space of embeddings for that to be a fiber sequence, but I will not worry about it here.

We would like to argue as follows: since $K\#W_g$ has finite fundamental group, Triantafillou's result (see the Remark at the end of Section 1) implies that the mapping class group of $K\#W_g$ is commensurable with an arithmetic group. Therefore A would be a finitely generated abelian group (by a result of Malcev) and therefore $\pi_0(\mathrm{Emb}^{\cong}(K\#W_g))$ would be a group of type F_∞ (Because in an extension of groups, if the leftmost and middle terms are of type F_∞ so must be the rightmost term). But this argument breaks here: Sullivan and Triantafillou's arithmeticity results are unknown for manifolds with boundary. This is very sad. So we had to tweak the argument to avoid having to use such inexistent arithmeticity result (which by the way, it would be wonderful if somebody proves it). I will not explain how we did it here, though. But you should not feel disappointed, I have just explained to you the "backbone" of the proof. The other tricks and details will be revealed only to true manifold topologists.