TALBOT 2023: Computations in stable motivic homotopy theory

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Overview

Motivic homotopy theory involves the study of algebraic varieties and schemes using methods not only from algebraic geometry, but also from algebraic topology [Voe98]. A basic idea is to use techniques from homotopy theory, such as the construction of homotopy groups and the use of homotopy invariants, to investigate topological and geometric properties of these varieties. It was developed in the late 20th century as a way to extend classical homotopy theory, which studies topological spaces up to continuous deformations, to the realm of algebraic geometry. The goal of the workshop is to discuss "Why, what, and how" on specific computations in stable motivic homotopy theory.

The *stable stems* or the stable homotopy groups of spheres

$$\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/24, 0, 0, \mathbb{Z}/2, \mathbb{Z}/240, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^3, \mathbb{Z}/6, \mathbb{Z}/504, 0, \mathbb{Z}/3, (\mathbb{Z}/2)^2, \mathbb{Z}/408 \oplus \mathbb{Z}/2, \ldots$$

carry a rich and mysterious structure and have been studied extensively in homotopy theory [Tod62], [Hop02]. For example, homotopy classes of maps $S^n \to S^n$ between n-dimensional spheres are classified by their degree. This leads to the identification of the zeroth stable stem with the integers. These invariants remain a measuring stick for our understanding of algebraic topology.

More nuanced invariants, depending on arithmetic of the base field F, enter the study of stable motivic homotopy groups. One of the fundamental problems in motivic homotopy theory over F is to calculate the homotopy groups $\pi_{t,w}\mathbf{1}$ of the motivic sphere spectrum $\mathbf{1}$. Here the integers $t, w \in \mathbb{Z}$ refer to the topological degree and the weight, respectively. In a precise sense, these groups are the universal motivic invariants because the motivic sphere is the unit for the tensor product on motivic spectra. All the relations witnessed in the graded ring $\pi_{*,\star}\mathbf{1}$ hold in every other theory representable in the stable motivic homotopy category, such as algebraic cobordism, algebraic and hermitian K-theory, motivic cohomology, and higher Witt theory. In an influential result, Morel identified the endomorphism ring of the motivic sphere with the Grothendieck-Witt ring $GW(F)$ that encodes the quadratic form data of F; a surprising fact since the construction of motivic homotopy theory does not involve quadratic forms.

By work of Morel [Mor06], the group $\pi_{t,w}\mathbf{1} = 0$ if $t < w$ and $\bigoplus_{n\in\mathbb{Z}} \pi_{n,n}\mathbf{1}$ is isomorphic to the Milnor-Witt K-theory $\mathbf{K}_*^{MW}(F)$ of F. Over the complex numbers, Levine [Lev14] showed that the Betti realization functor witnesses all the classical stable homotopy groups as the weight zero part of the corresponding groups of the motivic sphere. The key inputs enabling the connection between motivic homotopy groups and quadratic forms are the resolutions of the Bloch-Kato, and Milnor conjectures on Galois cohomology and quadratic forms in [Voe03], [OVV07], and [Voe11].

Röndigs-Spitzweck-Østvær [RSØ19] continued these calculations to the Milnor-Witt 1-stem $\bigoplus_{n\in\mathbb{Z}} \pi_{n+1,n}\mathbf{1}$. Their approach makes a systematic analysis of the slice spectral sequences for the sphere and hermitian K-theory [Voe02], [RØ16] (the latter reference gives an alternative proof of Milnor's conjectures on quadratic forms). To "turn the pages" in the slice spectral sequence for the sphere, one calculates sufficiently many differentials to deduce that it collapses at its E^2-page, at least as far as the Milnor-Witt 1-stem is concerned. A case-by-case analysis achieves this, which combines Voevodsky's description of the motivic Steenrod algebra [Voe03], [HKØ17], with the observation that the slice differentials induce graded Milnor K-theory module maps [Mil70].

In particular, [RSØ19] verifies Morel's π_1-conjecture by showing there is a short exact sequence

$$0 \to \mathbf{K}_2(F)/24 \to \pi_{1,0}\mathbf{1} \to F^\times/2 \oplus \mathbb{Z}/2 \to 0 \tag{0.1}$$

This is a key ingredient in Asok-Fasel-Williams' proof of Suslin's conjecture in degree five on the image of Quillen K-theory in Milnor K-theory [AFW20], and also in Röndigs' proof of the same conjecture in degree four [Rön21]. Subject to a passage from stable to unstable homotopy groups, a geometric application of (0.1) is the Asok-Fasel solution of Murthy's conjecture on splitting line bundles off of rank d vector bundles with trivial top Chern class over smooth affine F-varieties of dimension d. Here F is an algebraically closed field of characteristic zero.

The Milnor-Witt 2-stem $\bigoplus_{n\in\mathbb{Z}} \pi_{n+2,n}\mathbf{1}$ is identified in [RSØ21], revealing new relations between algebraic cycles, algebraic vector bundles equipped with quadratic forms, and stable motivic stems. Moving up from the Milnor-Witt 1-stem to the 2-stem increases the computational complexity, which is rooted in the problem of controlling mod-n motivic cohomology groups as n increases. Owing to a comparison between the motivic sphere and very effective hermitian K-theory, based on a filtration defined in [SØ12], the paper [RSØ21] strengthens the main result in [RSØ19].

General organization

Besides the introductory lecture held by Oliver Röndigs and the concluding lecture held by Paul Arne Østvær, there are about 17 lectures listed, to be delivered by the graduate student/postdoctoral participants.

1 Day 1: Preliminaries

1.1 Lecture 1: Overview

(By the organizers). This lecture will be an overview of the whole endeavor.

1.2 Lecture 2: Construction and properties of the stable homotopy category over a field

References: Voevodsky's ICM lecture [Voe98], Morel's Trieste Lecture Notes [Mor03], [AE17]
Dependencies: No

1. Definition of the stable motivic homotopy category over a base, fundamental properties, homotopy purity, functoriality, topological realizations

1.3 Lecture 3: Filtrations on the stable motivic homotopy category

References: [Voe02], [SØ12], [Bac17], [BKWX22] Dependencies: 1.2

1. Spheres, connectivity, effectivity, Chow filtration
2. Spectral sequences

1.4 Lecture 4: Motivic cohomology and motives

References: [Voe98], perhaps a few parts of [MVW06], [Voe11] Dependencies: Lecture 1.2

1. Describe the motivic spectrum $H\mathbb{Z}$
2. Introduce Voevodsky's derived category of motives and its relation to SH
3. State the Milnor conjecture on quadratic forms and Bloch–Kato conjectures

2 Day 2: Identifying some fundamental invariants

2.1 Lecture 5: The motivic Steenrod algebra

References: [Voe10], [HKØ17], perhaps [Spi18]
Dependencies: 1.2, 1.4

1. Give a description of the motivic Steenrod algebra and the relation with the ordinary Steenrod algebra

2.2 Lecture 6: Sample computations with the motivic Adams spectral sequence

References: [DI10], [WØ17] Dependencies: 1.2, 1.3, 1.4, 2.1

1. Setting up the motivic Adams spectral sequence
2. Sample computations for simple fields: Complex numbers, finite fields

2.3 Lecture 7: The zeroth Milnor-Witt stem

Morel's Trieste Lecture Notes [Mor03], [Mor04], [Mor12], [GSZ16] Dependencies: 1.2

1. Describe Milnor–Witt K-theory and its basic properties (fiber product presentation)
2. State Morel's computation of zeroth homotopy sheaf of motivic sphere

2.4 Lecture 8: Background on MU and BP

References: Complex cobordism and stable homotopy groups of spheres [Rav86], [Zah72]
Dependencies: Topology
Contents:

1. Describe the complex cobordism spectrum MU, its coefficient ring and cooperations; do the same for BP
2. Set up the Adams-Novikov spectral sequences based on MU and BP
3. Give sample computation regarding the α-family and the image of J

3 Day 3: Computing slices

3.1 Lecture 9: More on cohomology theories and SH

References: [Voe98], Open problems in motivic stable homotopy theory [Voe02], [Hor05], [RØ16]
Dependencies: 1.2, 1.3, 1.4

1. Describe algebraic K-theory spectrum KGL, algebraic bordism MGL, Grothendieck–Witt spectrum KQ
2. The motivic Hopf map and relation between KQ and KGL
3. Describe the slice filtration on SH

3.2 Lecture 10: Warmup – slice computations

References: [Voe02], [Spi10], [RØ16]
Dependencies: 1.2, 1.3, 1.4, 2.4

1. Slices of algebraic K-theory KGL, algebraic bordism MGL
2. Slices of Grothendieck–Witt theory KQ and Witt theory KQ$[\eta^{-1}]$

3.3 Lecture 11: The slices of the sphere

References: [Voe02], [Lev14], [Lev15], [RSØ19]
Dependencies: 1.2, 1.3, 2.3, 3, 3.1, 3.2

1. State Voevodsky's conjecture on slices of the sphere spectrum $\mathbf{1}$
2. Sketch the proof in Levine's paper

3.4 Lecture 12: Convergence of the slice filtration

References: [Voe02], [Lev13], [RSØ19], [AM17]
Dependencies: 1.3, 3.3

1. Discuss convergence problems
2. Sketch the proof of convergence of the slice filtration over fields of finite cohomological dimension
3. Generalize/specialize to cellular spectra of finite type and arbitrary fields, using [AM17]

4 Day 4: Differentials and vanishing results

4.1 Lecture 13: The first slice differential for the sphere

References: [RØ16], [RSØ19]
Dependencies: 2.1, 3.1, 3.2, 3.3

1. Describe first slice differential for $\mathrm{KGL}/2$, $\mathrm{KQ}[\eta^{-1}]$ and KQ
2. Give information on first slice differential for the sphere spectrum
3. Connect to Morel's Adams spectral sequence discussion

4.2 Lecture 14: Vanishing of some higher differentials for the sphere

References: [OVV07], [MS10], [RSØ19], [RSØ21]
Dependencies: 2.1, 3.4, 4.1

1. Relate Steenrod operations with Milnor K-theory
2. Describe exact sequence from [OVV07]
3. Sketch desired vanishing

4.3 Lecture 15: Vanishing results for the η-inverted sphere

References: [Rön19], [AM17], [BH20]
Dependencies: 1.2, 2.3, 3.1

1. Discuss connective version $\mathrm{KQ}[\eta^{-1}]_{\geq 0}$ of Witt theory
2. Morel's theorem implies connectivity of unit map $\mathbf{1}[\eta^{-1}] \to \mathrm{KQ}[\eta^{-1}]_{\geq 0}$
3. Sketch proof of vanishing of $\pi_1 \mathbf{1}[\eta^{-1}]$ and $\pi_2 \mathbf{1}[\eta^{-1}]$

5 Day 5: Getting things done

5.1 Lecture 16: Computation of the first and second Milnor-Witt stems

References: [RSØ19], [RSØ21]
Dependencies: Everything

1. Compute relevant parts of slice spectral sequence for **1**
2. Use convergence and vanishing to show what it computes

5.2 Lecture 17: Comparison with unstable homotopy groups

References: [AF15], [AWW17], [AFW20]
Dependencies: 5.1

1. Relate stable to unstable computations
2. Give applications to Murthy's and Suslin's conjectures

5.3 Lecture 18: Application to classical stable stems

References: [GWX21], [Isa19], [IWX20], [BKWX22]
Dependencies: 2.2, 2.4

1. Construct and analyze the motivic spectrum "Cofiber of τ" over $\mathbb{C}$
2. Indicate a generalization to other fields via the Chow t-structure (if time allows it)
3. Describe motivic homotopy theory over $\mathbb{C}$ as a deformation of classical homotopy theory
4. Deduce classical Adams differentials from motivic homotopy theory

5.4 Lecture 19: Conclusion

(By the organizers). Summary and open problems.

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