Let G be a finite group. If V is a complex vector space equipped with an action of G, we let $V^G \subseteq V$ denote the subspace of *invariants* (consisting of those elements $v \in V$ satisfying $g(v) = v$ for all $g \in G$), and $V_G \twoheadleftarrow G$ the quotient space of *coinvariants* (defined as the quotient of V by the subspace spanned by elements of the form $g(v) - v$). These vector spaces are canonically isomorphic via the *norm map*

$$\mathrm{Nm}_G : V_G \xrightarrow{\sim} V^G,$$

which carries an element $\overline{v} \in V_G$ represented by $v \in V$ to the sum $\sum_{g \in G} g(v)$. The inverse isomorphism is given by the composition

$$V^G \subseteq V \xrightarrow{1/|G|} V \twoheadrightarrow V_G$$

Note that it is essential here that the field $\mathbf{C}$ of complex numbers has characteristic zero, so that division by $|G|$ is well-defined. If we were to work instead with a representation V of G over a field k whose characteristic divides the order of G, then the norm map $\mathrm{Nm} : V_G \to V^G$ need not be an isomorphism.

The norm construction has a counterpart in homotopy theory. Given an action of G on a spectrum X, one can form a spectrum X^{hG} of *homotopy invariants* and a spectrum X_{hG} of *homotopy coinvariants*, which are again related by a norm map $\mathrm{Nm} : X_{hG} \to X^{hG}$. This map is generally not invertible. However, one has the following weaker result of Hovey-Sadofsky ([10]):

Theorem 0.0.1 (Hovey-Sadofsky). *Fix a prime number p and an integer $0 \leq n < \infty$, and let X be a $K(n)$-local spectrum equipped with an action of a finite group G. Then the norm map*

$$\mathrm{Nm} : X_{hG} \to X^{hG}$$

is a $K(n)$-local homotopy equivalence. In other words, it exhibits the homotopy invariant spectrum X^{hG} as the $K(n)$-localization of the homotopy coinvariant spectrum X_{hG}.

Remark 0.0.2. In the special case $n = 0$, the spectrum X is *rational* (that is, its homotopy groups are rational vector spaces). In this case, a homotopy inverse to the norm map can again be given by the composition

$$X^{hG} \to X \xrightarrow{1/|G|} X \to X_{hG},$$

where the middle map is defined as a homotopy inverse to the multiplication map $|G| : X \to X$. However, when $n > 0$, the map $|G| : X \to X$ is never invertible (unless $X = 0$ or $|G|$ is relatively prime to p), so the content of Theorem 0.0.1 is much more subtle.

Theorem 0.0.1 has subsequently been generalized in several ways:

- In [12], Kuhn proves an analogue of Theorem 0.0.1 in the setting of $T(n)$-local spectra (as opposed to $K(n)$-local spectra).
- In [9], Hopkins and Lurie prove a generalization of Theorem 0.0.1, replacing the notion of a spectrum with an action of a finite group G by the notion of a local system $\mathcal{L}$ of $K(n)$-local spectra over a space S which is *π-finite* (that is, having only finitely many homotopy groups, each of which is a finite group). In this case, they construct a canonical map $\mathrm{Nm} : \mathrm{hocolim}_{s \in S} \mathcal{L}_s \to \mathrm{holim}_{t \in S} \mathcal{L}_t$, which is again a $K(n)$-local homotopy equivalence (this reduces to Theorem 0.0.1 in the special case where $S = BG$ is the classifying space of a finite group).
- In [2], Carmeli, Schlank, and Yanovski give a somewhat different proof of a stronger theorem, extending the Hopkins-Lurie result to the setting of $T(n)$-local spectra.

The goal of this workshop is to outline the proofs of these results and some of the surrounding mathematical ideas, and to describe some applications.

Lecture 1: Overview. Recall the construction of Dijkgraaf-Witten theory over the complex numbers, and note that it is essential to the construction that for any complex representation V of a finite group G, the subspace of invariant vectors $V^G \subseteq V$ can be identified with the space of coinvariants V_G. Recall the Tate construction in homotopy theory, the work of Hovey-Sadofsky and Kuhn, and give an informal summary of what the main theorem will say. Talk will be given by one of the mentors (but a good reference is [7]).

Lecture 2: Review of chromatic homotopy theory. This lecture should recall some of the main actors in chromatic homotopy theory, such as the theory of $K(n)$-local and $T(n)$-local spectra. For this there are lots of references one could use; for example [11]

Lecture 3: This lecture should review the Tate construction and show that it vanishes in the $T(n)$-local setting. Here there is a very nice short proof due to Clausen and Mathew ([3]) one can follow. The argument makes use of the Kahn-Priddy theorem, which might be worth reviewing as well. Older references include [12] and

Lecture 4: This lecture should review the general categorical framework surrounding ambidexterity. It should define local systems on a space with values in an ∞-category $\mathcal{C}$, the pullback functor f^* associated to a map of spaces $f : X \to Y$, and its adjoints $f_!$ and f_* (when they exist). It should explain the inductive strategy for constructing a norm map $\mathrm{Nm} : f_! \to f_*$, and define the notion of a $\mathcal{C}$-ambidextrous map of spaces $f : X \to Y$. The main reference is section 4 of [9] (but the lecture should set things up in less generality; it is probably better to avoid the general notion of Beck-Chevalley fibration).

Lecture 5: This lecture should introduce the notion of m-semiadditivity for an ∞-category $\mathcal{C}$. It should explain what it means concretely for small values of m, and that presentable m-semiadditive ∞-categories can be understood as presentable ∞-categories that are modules over a certain ∞-category of correspondences. The main reference is [6].

Lecture 6: This lecture should state the main theorem (the m-semiadditivity of $T(n)$-local stable homotopy theory for all n) and provide an outline of the proof, following the method of [2, section 1.3]. Integration of maps and some basic properties should be explained as in [2, sections 1.2,3.1,3.2]

Lecture 7: This lecture should explain the construction of the δ-power operation, following [2, section 4].This includes the basic algebraic theory of additive p-derivations, the construction of δ and the proof that δ is an additive p-derivations. The May nilpotence conjecture should be deduced. It is also possible to relate δ to Witt-vectors.

Lecture 8: This lecture should explain the "bootstrap machine" (Theorem 4.3.10) of [2] using the notion of amenability [2, sections 2.4,4.3]. Some emphasis should be put on Theorems 2.4.4 and 2.4.5.

Lecture 9: This lecture should complete the proof of $T(n)$-local ambidexterity, assuming that the Morava K-theory of Eilenberg-MacLane spaces is concentrated in even degrees. It should also explain how to extract some related results (such as the $T(n)$-local acyclicity of the Eilenberg-MacLane space $K(\mathbf{F}_p, n{+}1)$). Following [2, section 5] the lecture should conclude with the notion of a weak ring and theorems 5.4.11. and . 5.4.14.

Lecture 10: This lecture should give an overview of Dieudonné theory, which provides a classification of commutative formal groups (and commutative finite flat group schemes) over a perfect field k of characteristic $p > 0$. Here there are lots of references one could use (for example [4]).

Lecture 11: This lecture should explain how to classify bilinear maps between commutative finite flat group schemes (over a perfect field k) in terms of their Dieudonné modules, following Section 1 of [9] (other relevant references include [5] and [1]. It should then formulate the Ravenel-Wilson theorem (calculating the Morava K-theory of Eilenberg-MacLane spaces $K(\mathbf{Z}/p^t\mathbf{Z}, m)$) in terms of Dieudonné modules, following Section 2 of [9] (and [1]).

Lecture 12: This lecture should prove the result of Ravenel-Wilson which was stated in the previous lecture. The original reference is [15] (at odd primes), but a simplified approach is given in Section 2 of [9]. Warning: this is probably the hardest talk of the week.

Lecture 13: This lecture should use the Ravenel-Wilson calculation to deduce a description of the Morava E-theory of Eilenberg-MacLane spaces (using algebro-geometric language), following Section 3 of [9] (another reference is [14]). Probably it would be best to explain this only in the case where p is an odd prime, to avoid some unnecessary complications. (Note: we can probably omit this lecture if we need to.)

Lecture 14: This lecture should sketch the proof of Theorem 5.4.2 of [9] (developing other ideas from Section 5 as necessary). Roughly speaking, this asserts that if X is a p-finite space with vanishing homotopy in degrees $> n$, then the based loop space $\Omega(X)$ behaves like a "unipotent" group from the perspective of $K(n)$-local homotopy theory (meaning that its representations, or equivalently local systems on X, can be recovered from their global sections).

Lecture 15: This lecture should introduce the notion of an oriented p-divisible group and the associated tempered cohomology theory, following Section 4 of [13]. Here it would probably be best to view an oriented p-divisible group over a commutative ring spectrum A as a functor

$$\{\text{Classifying spaces of finite abelian } p\text{-groups}\} \to \{\text{Finite flat } A\text{-algebras}\}$$

with certain properties, and to regard tempered cohomology as an invariant of spaces rather than orbispaces. One can then prove Theorem 1.1.17 (which is immediate from the definition) and formulate Theorem 1.1.19/Theorem 4.7.1 (which is not at all obvious).

Lecture 16: This lecture should explain the character isomorphisms for tempered cohomology (following section 4.3 of [13]) and explain how to use them (together with Theorem 4.7.1) to recover various results like the character theory of finite groups, the character theory of Hopkins-Kuhn-Ravenel ([8]), and the transchromatic character theory of Stapleton ([16]). The relevant deductions are explained in Section 4 of [13] (and are relatively formal assuming Theorem 4.7.1).

Lecture 17: This lecture should sketch the proof of Theorem 4.7.1 of [13]. It should introduce the theory of tempered local systems (following Section 5 of [13]), state the tempered ambidexterity theorem (Theorem 7.2.10 of [13]), and explain how to use it to deduce Theorem 4.7.1. It can then sketch the strategy for proving ambidexterity (though probably not in too much detail, given time constraints), following section 7 of [13].

Lecture 18: This lecture will be given by one of the mentors to wrap things up. This would be a good place for Tomer to discuss some of his unpublished work following up on these ideas. Alternatively, Jacob could discuss connections with (2)-equivariant elliptic cohomology.

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