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Tempered cohomology (Elliptic III).

Constr: X - finite CW-complex

$KU^0(X) :=$ Grothendieck grp of $V \to X$ $= \mathbb{Z}[V \to X] / [V \oplus W] \sim [V] + [W]$

This extends to a 2-periodic coh. theory rep by KU.

$V, W \to X$
$\mapsto V \otimes W$ $\rightsquigarrow$ KU is an E_∞ ring spectrum.

G finite group $\quad X$ - finite CW cpx $\circlearrowleft G$

$$KU_G^0(X) = Gr\left(\begin{matrix} V & \to & X \\ \circlearrowleft & & \circlearrowleft \\ G & & G \end{matrix} \right).$$

Ex: $KU_G^0(pt) = Gr(V \circlearrowleft G) = Rep(G)$, described by char theory.

Thm: Relation?

$KU_G^0 \ni [V \to X] \rightsquigarrow [V/G \to X/G] \in KU_G^0(X_{hG})$.

Thm: (Atiyah-Segal).

$Rep(G) \simeq KU_G^0(pt) \to KU^0(BG)$

is a completion at the aug. ideal.

G - p-group $\qquad I_G \subseteq {}_p\mathrm{Rep}(G)$.

$\to \quad KU^0_G(X) \to KU^0_G(X_{hG})$ an iso. after p-completion.

$KU^\wedge_p = E_1 \qquad$ Q: ~~Is this~~ Are these phenomena captured by chromatic homotopy?

Idea (Lurie):

1) The equivariant generalization of KU is determined by its chromatic structure (the Quillen fg).

2) ~~Any~~ ~~To extend any E~~ To give an equivariant gen. of a coh. theory rep. by an E_∞-ring A is the same as to choose an (oriented) p-div. group G/A.

Recall: A p-div. group of height n over a ring k is a formal group scheme which "looks like" $(\mathbb{Q}_p/\mathbb{Z}_p)^{\times n}$.

1) $\underset{\mathrm{CAlg}_A}{\overset{\in}{B}} \to \mathrm{Hom}(\mathbb{Z}/p^k, G(B))$ is rep. by a flat free finite group scheme

2) $G \xrightarrow{p} G$ is an epimorphism.

We'll generalize this in two diff. ways.

1) Replace ku with a general E_∞-ring.

2) Remove dependence on the prime.

Notation: P = primes = $\{2, 3, \dots\}$.

Def: If A is an E_∞ ring, a ~~p-divisible group~~ functor $\mathcal{O}_{\mathbb{G}}: \mathrm{Ab}^{fin}_{\{p\}} \to \mathrm{CAlg}_A$ is p-divisible if:

1) $\mathcal{O}_{\mathbb{G}}$ preserves coproducts (in particular, $\mathcal{O}_{\mathbb{G}}(M)$ is a coproduct object in Hopf algebras) and $\mathcal{O}_{\mathbb{G}}(0) = A$).

2) for every ses. $0 \to M' \to M \to M'' \to 0$.

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{G}}(M'') & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{G}}(M) & \longrightarrow & \mathcal{O}_{\mathbb{G}}(M'') \end{array}$$

is a pushout with vertical arrows finite free.

A p-div. functor uniquely defines a p-div. group $\mathbb{G}$; encodes a family of p... [illegible]

Note: $\mathrm{Map}_{\mathrm{CAlg}_A}(\mathcal{O}_{\mathbb{G}}(M), B) \simeq \mathrm{Hom}_{\mathbb{Z}}(M^\vee, \mathbb{G}(B))$.

Ex: Constant $M \mapsto A^{M^\vee}$.

Ex: Multiplicative $M \mapsto A \otimes \Sigma^\infty_+ M$.

(~~Hom~~ $\mathrm{Map}_{\mathrm{CAlg}_A}(A \otimes \Sigma^\infty_+ M, B) \simeq \mathrm{Hom}_{E_\infty}(M, GL_1(B))$.

Ex: $K(n)$-local A ($A = E_n$). 2-periodic.

Quillen

$$M \mapsto A^{BM^\vee}.$$

(~~$\pi_* A^{\mathbb{Z}/p}$~~ (Spec $\pi_0 A^{\mathbb{Z}/p} \simeq$ Spf $A^{\mathbb{CP}^\infty}[p]$)).

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To get an eqv. genuine functor, we need further data than a P-dv. functor (an orientation).

<u>Notation</u>: $\mathcal{J}^{ap} \subseteq \mathcal{S}_{pc}$

subset of classifying spaces of f abelian grs.

For a P-dv. $\mathcal{O}_G : AG \to CAlg_A$

Prop: The following choice of data are eqv.

1) a nat. transf. $A^{BM^\vee} \to \mathcal{O}_G(M)$.

2) a factorization.

$$\begin{tikzcd} AG \arrow[r] \arrow[d, "M \mapsto BM^\vee"'] & CAlg_A \\ \mathcal{J}^{ap} \arrow[ur, dashed] & \end{tikzcd}$$

We call this data an <u>orientation</u> of P-dv. $\mathcal{O}_G$.

<u>Ex</u>: A $K(n)$-local 2-periodic.

$\mathcal{O}_G(M) \simeq A^{BM^\vee}$. identity.

<u>Ex</u>: $A = k$ discrete ring.

A^{BA} $\mathcal{O}_G(M)$ discrete for any M. $\to$ unique map

$A^{BM^\vee} \to (A^{BM^\vee})_{\geq 0} \to \mathcal{O}$

<u>Ex</u>: (Oriented P-dv. group on KU).

$\mathcal{J} \ni S \to \mathrm{Fun}(S, \mathrm{Vect}_{\mathbb{C}}^{\simeq})$

$ku(S) = $ Group completion $(\mathrm{Fun}(S, \mathrm{Vect}_{\mathbb{C}}^{\simeq}))$.

$KU(S) \simeq ku(S) \underset{ku(pt)}{\otimes} KU$.

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What's the underlying p-div. group?

$$M \to \pi_0 KU(B\check{M}) \simeq \mathrm{Rep}(\check{M}) \simeq \mathbb{Z}[M].$$

$$\mathbb{C}^\times \leftarrow \lambda : \check{M} \to \mathbb{Q}/\mathbb{Z} \to U(1)$$

Thus, the underlying P-div. group over $\pi_0 KU$ is the multiplicative one $\leadsto$ also true over KU.

Our eqv. coh. theory will take equivariant input.

Def: An orbispace $\mathcal{X}$ is a functor
$$\mathcal{X} : \mathcal{T}^{op} \to \mathrm{Spc}.$$
We denote the ∞-cat of orbispaces by $\mathcal{OS}$.

Note.. **Ex:** X ~~-space~~. Any space can be identified with orbispace.

$$\mathcal{X}(S) = \mathrm{Map}(S, X) \simeq X^S.$$ Both fully faithful.

This functor has a left adjoint. **Ex:** $X(S) = X$.

(a Orbispace quotient).

Ex: X - G-space (e.g. a CW-cpx with a G-action).

$$X \mapsto X /\!/ G = \varinjlim_{H^G \to X.} \left(\tfrac{G}{H}\right)_{hG} \simeq \varinjlim_{H \to G.} BH$$

H-orbifold

Note: If G is abelian, then $X \mapsto X/\!/G$ gives a fully faithful functor

$$\mathcal{OS}_G\text{-}\mathrm{Spc} \to \mathcal{OS}\mathrm{pc}/_{BG}.$$

Def. (Tempered cohomology).

A E_∞-ring, $\mathbb{G}: \mathcal{T}^{op} \to \mathrm{CAlg}_A$ oriented P-div. functor.

The tempered cohomology cochains funct.

$$A_{\mathbb{G}}^{(-)} : \mathcal{OS} \to \mathrm{CAlg}_A$$

is the unique ext of $\mathcal{O}_{\mathbb{G}}$ which takes colimits of orbispaces to limits.

Notation: $A_{\mathbb{G}}^*(X) = \pi_{-*} A_{\mathbb{G}}^{X}$.

Ex (Equivariant K-theory).

$A = KU$ $\quad$ $\mathbb{G} = \mu_\infty$. multiplicative.

$$KU^*_{\mu_\infty}(G/H // G) = KU^*_H.$$

If G is abelian, then G/H generates G-Spc under colimits and

$$KU^*_{\mu_\infty}(X // G) = KU^*_G(X).$$

(In fact, this is true when G is not abelian.)

Thm / Cor. A - even periodic $K(n)$-local. $\mathbb{G} = \mathbb{G}_A^{\mathbb{Q}}$.

For any orbispace X

$$A_{\mathbb{G}}^*(X) = A^*(|X|).$$

Proof. True when $X = B\hat{M}$ by definition + both sides take colimits to limits.

Thm: Let $A \to B$ be a flat map of p-local E_∞-rings, $\mathbb{G}$ an oriented p-adic grp over A and $\mathbb{G}_B := \mathbb{G} \otimes_A B$ the extension to B. Then for any finite H-space X we have

$$B^\alpha_{\mathbb{G}_B}(X/\!/B) = B^\alpha_\ast \otimes_{A^\alpha} A^\alpha_{\mathbb{G}}(X/\!/\mathbb{G}).$$

takes colimits to limits.

($\otimes$ preserves all colimits but not all cond limits unless B is fte.