<u>Goal</u>: Define a tensor product on Hopf-algebras and on Dieudonné modules and show that

$$\mathrm{DM} : \mathrm{Hopf}_{\kappa}^{c} \longrightarrow \mathrm{Mod}_{D_\kappa} \quad \text{is symmetric monoidal.}$$

Then state the Ravenel-Wilson theorem using this tensor product

I <u>The tensor product on Hopf algebras</u>

Fix κ a perfect field of characteristic $p>0$.

<u>Def</u> A commutative formal group over κ is a functor $G : \mathrm{CAlg}_{\kappa}^{fd} \longrightarrow \mathrm{Ab}$ that preserves finite limits.

<u>Remark</u>:

Formal group laws $\longrightarrow$ Formal groups

$$F(x,y) \in \kappa[\![x,y]\!] \longmapsto \begin{array}{l} G_F : \mathrm{CAlg}_{\kappa}^{fd} \longrightarrow \mathrm{Ab} \\ G_F(A) = (\mathrm{Nil}(A), +_F) \end{array}$$

Here $\mathrm{Nil}(A) \subset A$ is the ideal of nilpotent elements and $a +_F b := F(a,b) \in \mathrm{Nil}(A)$.

<u>Ex</u> Additive FGL $A \mapsto (\mathrm{Nil}(A), +)$ Multiplicative: $A \mapsto (1 + \mathrm{Nil}(A), \cdot)$

Lemma: $\mathrm{Spf}(-)^\vee : \mathrm{Hopf}_k \to \mathrm{CFG}_k$ is an equivalence.

$$(\mathrm{Spf}\, H^\vee)(A) = \mathrm{GLike}(H \otimes A)$$

Proof idea
- $\mathrm{Hopf}_k = \mathrm{Ab}(\mathrm{CoAlg}_k)$
- $\mathrm{CoAlg}_k = \mathrm{Ind}(\mathrm{CoAlg}_k^{fd}) = \mathrm{Fun}^{lex}(\mathrm{CoAlg}_k^{fd,op}, \mathrm{Set})$
- $\mathrm{CoAlg}_k^{fd} \simeq (\mathrm{Alg}_k^{fd})^{op}$ $\square$

Remark Limits in CFG_k are computed pointwise in $\mathrm{Fun}(\mathrm{Alg}_k^{fd}, \mathrm{Ab})$

Eg. $(G \times G')(A) = G(A) \times G'(A)$

in Hopf: $H \otimes_k H'$ is the categorical product (not the "tensor product")

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Now define the tensor product on CFG_k through bilinear pairings.

Def: A bilinear pairing is a natural transformation $\mu : G \times G' \to G''$ in $\mathrm{Fun}(\mathrm{CAlg}_k^{fd}, \mathrm{Set})$ such that for all $A \in \mathrm{CAlg}_k^{fd}$

$$\mu_A : G(A) \times G'(A) \longrightarrow G''(A) \quad \text{is bilinear.}$$

i.e. $\mu_A(x+y, z) = \mu_A(x,z) + \mu_A(y,z)$ and $\mu_A(x, z+w) = \dots$

Lemma There is a universal bilinear pairing $G \times G' \to G \otimes G'$

This defines a symmetric monoidal structure on CFG_k.

Proof idea: Use $\mathrm{CFG}_k \underset{\perp}{\rightleftarrows} \mathrm{Fun}(\mathrm{CAlg}_k^{fd}, \mathrm{Ab})$ and define

$$G \otimes G' = L(\text{pointwise tensor product}).$$ $\square$

This also defines $(H, H') \longmapsto H \boxtimes H'$ on Hopf_k

Properties:

1) $\mathrm{Sym}(C) \boxtimes \mathrm{Sym}(C') \cong \mathrm{Sym}(C \otimes C')$ for C, C' coalgebras
2) There is a canonical coalgebra map $H \otimes H' \longrightarrow H \boxtimes H'$ which extends to a Hopf algebra map $\mathrm{Sym}(H \otimes H') \to H \boxtimes H'$
3) $H \boxtimes H' = \mathrm{Sym}(H \otimes H')/\text{relations}$

$$1 \boxtimes x = \varepsilon(x) \cdot 1 \qquad (x \cdot y) \boxtimes z = \sum_i (x \boxtimes z_i^{(1)}) \cdot (y \boxtimes z_i^{(2)})$$

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II Witt vectors and Dieudonné modules

Recall: $Wt_{Big} = \mathbb{Z}[c_1, c_2, \ldots]$ represents $R \mapsto (1 + tR[[t]], \cdot)$

Other coordinates: $\underset{\text{"ghost"}}{\mathbb{Z}[w_1, w_2, \ldots]} \hookrightarrow \underset{\text{"Witt"}}{\mathbb{Z}[a_1, a_2, \ldots]} \cong \mathbb{Z}[c_1, c_2, c_3, \ldots]$

$Wt_n := \mathbb{Z}[a_1, a_p, \ldots, a_{p^{n-1}}]$ and $Wt_n^k = Wt_n \otimes_{\mathbb{Z}} k$

The Dieudonné module $DM(H) = \operatorname{colim}_{n \to \infty} \mathrm{Hom}_{\mathrm{Hopf}_k}(Wt_n^k, H)$ is a module over $D_k = W(k)[F, V] / _{FV = p,\ F\lambda = \varphi(\lambda)F,\ \lambda V = V\varphi(\lambda)}$

Thm This defines a fully faithful functor:

$$DM : \mathrm{Hopf}_k^c \longrightarrow \mathrm{Mod}_{D_k}$$

with image the V-nilpotent D_k-modules.

Note: there are also versions of this for non-connected Hopf algebras

$$\mathrm{Hopf}_k^{p\text{-nil}} \hookrightarrow \mathrm{Mod}_{D_k}$$

where H is p-nilpotent if $H = \bigcup_{n\geq 0} H[p^n]$ $\rightarrow$ $\ker [p^n]: H \to H$

The image consists of those $M \in \mathrm{Mod}_{D_k}$ where for each $x \in M$ the $W(k)$-submodule spanned by $x, Vx, V^2x, \dots$ has finite length.

<u>Ex</u> A p^n-torsion abelian group

$$DM(k[A]) = W(k) \otimes_{\mathbb{Z}} A \quad \text{with} \quad \begin{array}{l} F(\lambda a) = p\,\varphi(\lambda)\,a \\ V(\lambda a) = \varphi^{-1}(\lambda)\,a \end{array}$$

<u>Def</u> The tensor product $\tilde{\otimes}$ on Mod_{D_k} is defined as follows:

$\mu: M \times M' \to M''$ is a pairing if:

1) it is $W(k)$-bilinear
2) $V\mu(x,y) = \mu(Vx, Vy)$
3) $F\mu(x, Vy) = \mu(Fx, y)$
4) $F\mu(Vx, y) = \mu(x, Fy)$

Let $M \times M' \to M \tilde{\otimes} M'$ denote the universal pairing.

Theorem For any two connected Hopf algebras H, H'

a) There is a natural map $\mu : DM(H) \times DM(H') \to DM(H \boxtimes H')$

b) μ is a pairing and hence factors $DM(H) \tilde{\otimes} DM(H') \xrightarrow{\alpha} DM(H \boxtimes H')$

c) The map α is an isomorphism

d) This makes $DM : (\mathrm{Hopf}_k^c, \boxtimes) \to (\mathrm{Mod}_{D_k'}, \tilde{\otimes})$ a s.m. functor.

To construct μ we will find $\iota_n : Wt_n^k \to Wt_n^k \boxtimes Wt_n^k$ and define:

$$\begin{array}{ccccc} \mathrm{Hom}(Wt_n^k, H) \times \mathrm{Hom}(Wt_n^k, H') & \xrightarrow{\boxtimes} & \mathrm{Hom}(Wt_n^k \boxtimes Wt_n^k, H \boxtimes H') & \xrightarrow{\iota_n^*} & \mathrm{Hom}(Wt_n, H \boxtimes H') \\ \| & & & & \| \\ DM(H)_n \times DM(H')_n & & \xrightarrow{\mu_n} & & DM(H \boxtimes H')_n \end{array}$$

This assembles to a well-defined $\mu : DM(H) \times DM(H') \to DM(H \boxtimes H')$

if
$$\begin{tikzcd} Wt_n^k \arrow[r, "\iota_n"] \arrow[d, "V"'] & Wt_n^k \boxtimes Wt_n^k \arrow[d, "V \boxtimes V"] \\ Wt_{n-1}^k \arrow[r, "\iota_{n-1}"] & Wt_{n-1}^k \boxtimes Wt_{n-1}^k \end{tikzcd}$$
commutes.

Lemma There is a unique Hopf algebra map $\iota : Wt_n \to Wt_n \boxtimes Wt_n$ satisfying $\iota(w_k) = \frac{w_k \boxtimes w_k}{k}$

Proof Rationally there is a unique algebra map $\iota^{\mathbb{Q}}: Wt_{Big}^{\mathbb{Q}} \to Wt_{Big}^{\mathbb{Q}} \boxtimes Wt_{Big}^{\mathbb{Q}}$

satisfying $\iota^{\mathbb{Q}}(\omega_n) = \frac{1}{n}(\omega_n \boxtimes \omega_n)$.

This is also a coalgebra map:

$$\iota^{\mathbb{Q}}(\Delta\omega_n) = \iota^{\mathbb{Q}}(\omega_n \otimes 1 + 1 \otimes \omega_n) = \frac{1}{n}(\omega_n \boxtimes \omega_n) \otimes (1 \boxtimes 1) + \frac{1}{n}(1 \boxtimes 1) \otimes (\omega_n \boxtimes \omega_n)$$

$$n \cdot \Delta \iota^{\mathbb{Q}}(\omega_n) = \Delta(\omega_n \boxtimes \omega_n) = (\Delta\omega_n) \boxtimes (\Delta\omega_n) = (\omega_n \otimes 1 + 1 \otimes \omega_n) \boxtimes (\omega_n \otimes 1 + 1 \otimes \omega_n)$$

$$= (\omega_n \boxtimes \omega_n) \otimes (1 \boxtimes 1) + \underbrace{(\omega_n \boxtimes 1) \otimes (1 \boxtimes \omega_n) + (1 \boxtimes \omega_n) \otimes (\omega_n \boxtimes 1)}_{= 0 \text{ because } \omega_n \boxtimes 1 = \varepsilon(\omega_n) = 0} + (1 \boxtimes 1) \otimes (\omega_n \boxtimes \omega_n)$$

To get the integral map:

Fact $$Wt_n \boxtimes Wt_n \xrightarrow{\cong} (Wt_n^{\mathbb{Q}} \boxtimes Wt_n^{\mathbb{Q}}) \cap (Wt_{Big} \boxtimes Wt_{Big}) \subset Wt_{Big}^{\mathbb{Q}} \boxtimes Wt_{Big}^{\mathbb{Q}}$$

So it remains to check that $\iota^{\mathbb{Q}}(Wt_n \boxtimes Wt_n)$ is integral.

This is done by some clever power series manipulation "$\square$"

One checks $\iota_{n+1} \circ V = (V \boxtimes V) \circ \iota_n$ to prove a)

Proof that $\mu: DM(H) \times DM(H') \to DM(H \boxtimes H')$ is a pairing

- $W(k)$ bilinearity follows from definition of the action.
- $V\mu(x,y) = \mu(Vx, Vy)$ follows from the above
- $F\mu(x, Vy) = \mu(Fx, y)$: It suffices to check that $(Vx \boxtimes y)^p = x \boxtimes y^p$ holds in $Wt_n^k \boxtimes Wt_n^k$

$\square$

Proof that $\mu: DM(H) \tilde{\otimes} DM(H') \longrightarrow DM(H \boxtimes H')$ is an iso $\forall H, H'$

Both sides preserve small colimits in H and in H'.

$\hookrightarrow$ suffices to check on $H = W t_n^K$ and $H' = W t_m^K$

as the $W t_n^K$ generate Hopf_K^c under colimits

Next, use the cofiber sequence $K \to W t_1^K \to W t_n^K \to W t_{n-1}^K \to K$

and a 5-lemma argument to reduce to the case $n=1$ and $m=1$.

Now compute

$$\begin{array}{ccccc} DM(K[c_1]) & \tilde{\otimes} & DM(K[c_1]) & \longrightarrow & DM(\underbrace{K[c_1] \boxtimes K[c_1]}_{K[c]}) \\ \| & & \| & & \\ D_K/V & \tilde{\otimes} & D_K/V & \longrightarrow & D_K/V \end{array}$$

The D_K-module $D_K/V\langle x\rangle \tilde{\otimes} D_K/V\langle y\rangle$ is generated by $x \tilde{\otimes} y$

moreover $V(x \tilde{\otimes} y) = Vx \otimes Vy = 0$.

$\Rightarrow D_K/V \tilde{\otimes} D_K/V \cong D_K/V$ $\square$

d) To check that $\mu: DM(H) \tilde{\otimes} DM(H') \longrightarrow DM(H \boxtimes H')$

is compatible with the associator it suffices to check:

$$\begin{tikzcd} Wt_{Big} \arrow[r, "c"] \arrow[d, "c"'] & Wt_{Big} \boxtimes Wt_{Big} \arrow[d, "id \boxtimes c"] \\ Wt_{Big} \boxtimes Wt_{Big} \arrow[r, "c \boxtimes id"] & Wt_{Big} \boxtimes Wt_{Big} \boxtimes Wt_{Big} \end{tikzcd}$$

This can be done rationally

$\square$

III The Ravenel-Wilson theorem

Def A Hopf ring is a monoid object in $(\mathrm{Hopf}_\kappa, \boxtimes)$

This is the same as limit-preserving functors $\mathrm{CAlg}_\kappa^{\mathrm{fd}} \to \mathrm{ARing}$

Def For a Hopf algebra H the exterior Hopf ring on H is

$$\Lambda_\boxtimes(H) = \bigoplus_{n\geq 0} H^{\boxtimes n}_{\Sigma_n} = \kappa[t^{\pm 1}] \oplus H \oplus (H\boxtimes H)_{\Sigma_2} \oplus \dots$$

Write elements as $h_1 \circ \dots \circ h_n$ with $h_1 \circ h_2 = [-1]\, h_2 \circ h_1$

Ex: h_* a multiplicative homology theory satisfying Künneth
$(E_n)_{n\geq 0}$ an Ω-spectrum representing a ring spectrum E

Then $\bigoplus_{n\geq 0} h_*(E_n)$ is a Hopf ring.

Theorem $K(n)_* \, K(\mathbb{Z}/p^t, d)$ is concentrated in even degrees and $\bigoplus_{d\geq 0} K(n)_0 \, K(\mathbb{Z}/p^t, d)$ is the exterior Hopf ring on $K(n)_0 \, K(\mathbb{Z}/p^t, 1)$

$K(n)_0 \, K(\mathbb{Z},2) = K(n)_0 \, \mathbb{CP}^\infty \;\hat{=}\;$ formal group law of $K(n)$

$DM(K(n)_0 \, K(\mathbb{Z},2)) = D_\kappa / V^{n-1} - F = \mathbb{Z}_p \langle x, Vx, \dots, V^{n-1}x \rangle$

$DM(K(n)_0 \, K(\mathbb{Z}/p^t, 1)) = D_\kappa / V^{n-1} - F, p^t = \mathbb{Z}/p^t \langle x, Vx, \dots, V^{n-1}x \rangle$

$(FV = p,\; F = V^{n-1})$

In fact, the Dieudonné module of $K(n)_\circ K(\mathbb{Z}/p^t, d)$ is not so complicated:

$$DM(\Lambda_\otimes(K(n)_\circ K(\mathbb{Z}/p^t,1))) \cong \Lambda_{\tilde{\otimes}} DM(K(n)_\circ K(\mathbb{Z}/p^t,1))$$
$$\cong \Lambda_{\tilde{\otimes}} D_K/(V^{n-1}-F, p^t)$$

Prop $\Lambda_{\mathbb{Z}/p^t} D_K/(V^{n-1}-F, p^t) \xrightarrow{\cong} \Lambda_{\tilde{\otimes}} D_K/(V^{n-1}-F, p^t)$