TALBOT 2019: MODULI SPACES OF MANIFOLDS

This workshop is aimed at exposing recent progress in our understanding of the cohomology of moduli spaces of smooth manifolds, or in other words classifying spaces of diffeomorphism groups.

The focus of the first half of the workshop will be the method which arose from Madsen and Weiss' proof of the Mumford Conjecture [MW07], based on the theory of cobordism categories, fibrewise surgery, the group-completion theorem, and homological stability. A central role in most of what will be discussed will be played by the manifold

$$W_g^{2n} = \#^g S^n \times S^n,$$

the g-fold connected sum of $S^n \times S^n$, which should be considered the analogue in higher dimensions of the oriented genus g surface in dimension $2n = 2$. (In fact, the variant $W_{g,1} = W_g \setminus \mathrm{int}(D^{2n})$ will be most prominent.) However the techniques discussed in the first 9 talks apply quite generally, and Talk 10 will summarise the state of the art in this direction.

The second half of the workshop will give an overview of three adjacent pieces of mathematics: two applications of the theory developed in the first half of the workshop, and an analogous result. The first application is the recent theorem of Kupers [Kup17]—building on work of Weiss [Wei15]—establishing the cohomology of the classifying spaces of many diffeomorphism groups (including $B\mathrm{Diff}_\partial(D^{2n})$ for $2n \geq 6$) has finite type. The second is recent work of Botvinnik–Ebert–Randal-Williams [BERW17] on the non-triviality of spaces $\mathcal{R}^+(M)$ of Riemannian metrics of positive scalar curvature. The third is recent work of Berglund–Madsen [BM13, BM14] on classifying spaces for fibrations (as opposed to smooth fibre bundles).

Advice. To give a good talk you will need to be broadly familiar with the content of the other talks in the same series, so that you know what you should provide to speakers after you and what you can expect from speakers before you.

1. Cobordism category methods

This series of talks first aims to explain the main theorem of [GTMW09], which identifies the homotopy type of the category of d-dimensional cobordisms (possibly equipped with tangential structure). We will follow the proof of this theorem given in [GRW10].

Talk 1. Explain a definition of the cobordism category $\mathcal{C}_d$, a topological category whose objects are closed smooth $(d-1)$-manifolds and whose morphisms are d-dimensional cobordisms between them. See [GTMW09, Section 2.1] for a mostly self-contained definition (or [GRW10, Section

3.2] or [GRW14, Section 2.4]). Resist the temptation to discuss higher categories.

Define the nerve and the classifying space of a topological category $\mathcal{C}$, and the natural map $\mathrm{End}_{\mathcal{C}}(x) \to \Omega B\mathcal{C}$. Specialising to the cobordism category, explain why we get a map $B\mathrm{Diff}(W) \to \Omega B\mathcal{C}_d$ for any closed smooth d-manifold W.

Give a quick definition of $\Omega^\infty MTO(d)$ as a space, mentioning Grassmannians of d-planes in $\mathbb{R}^N$, and Thom spaces of vector bundles over them. For the purposes of the following talks, an explicit space-level definition as in [GTMW09, Section 1] is more useful than one-line definitions involving Thom spectra of virtual bundles. Then *state* the main theorem of [GTMW09] as the existence of a weak equivalence $\Omega B\mathcal{C}_d \simeq \Omega^\infty MTO(d)$.

If you have any time left, briefly mention "tangential structures"; for example, there is a version $\mathcal{C}_{SO(d)}$ of the cobordism category in which all manifolds are equipped with orientations.

Talk 2. This talk should introduce the *spaces of non-compact manifolds* denoted $\Psi_d(\mathbb{R}^N)$ in [GRW10]. You should not go into details with the topology on this space, but give an idea of what continuous maps $X \to \Psi_d(\mathbb{R}^N)$ looks like. Explain the weak equivalence from the Thom spaces appearing in Talk 1 to the space $\Psi_d(\mathbb{R}^N)$.

Then define the subspaces $\psi_d(n, 1) \subset \Psi_d(\mathbb{R}^n)$ consisting of manifolds bounded in all but one direction. State the weak equivalence $B\mathcal{C}_d \simeq \mathrm{colim}_n\, \psi_d(n, 1)$, and indicate how it is proved. You may not have time for a detailed proof, but try to emphasise the role played by Sard's theorem.

Talk 3. Define the spaces $\psi_d(n, k) \subset \Psi_d(\mathbb{R}^n)$ consisting of manifolds bounded in all but k directions. Define the map $\psi_d(n, k) \to \Omega\psi_d(n, k + 1)$, state that it is a weak equivalence for $k \geq 1$, and explain how the GMTW theorem follows.

Then give some details of the delooping theorem $\psi_d(n, k) \to \Omega\psi_d(n, k+1)$, as time permits. (The strategy can be paraphrased as follows: "to prove a weak equivalence of the form $X \simeq \Omega Y$, guess a suitable group-like topological monoid M and prove $X \simeq M$ and $Y \simeq BM$".)

Talk 4: Group completion. This talk should start with a statement of the group-completion theorem for topological monoids M, concerning the canonical map $M \to \Omega BM$. This map is a weak equivalence if and only if M is M is group-like, i.e. the induced monoid structure on $\pi_0(M)$ has inverses. The group-completion theorem for monoids ([MS76]) concerns its effect in homology when M is not group-like. The theorem has some assumptions, which should be explained in the talk. In particular it is sufficient that M to be homotopy commutative.

Monoids are categories with one object, and the group-completion theorem may be generalised to categories with more than one object. If $\mathcal{C}$ is a topological category then, in good cases, the homology of $\Omega B\mathcal{C}$ may be calculated as a filtered colimit of homology of morphism spaces $\mathcal{C}(x, y)$. (In this abstract setting, a key ingredient is the existence of a filtered diagram $f : J \to \mathcal{C}$ such that the functor $x \mapsto \operatorname{colim}_{j \in J} H_*(\mathcal{C}(x, f(j))$ sends all morphisms to isomorphisms.)

The group-completion theorem for categories was applied to cobordism categories by Tillmann ([Til97]) in the case of oriented 2-dimensional cobordisms, but the short exposition in [MT01, Section 2.2] may fit better with this series of talk. (The oriented 2-dimensional cobordism category is denoted $\mathcal{Y}$ in the latter reference.) The higher-dimensional case of the group-completion argument is carried out in [GRW17, Section 7.1], but for the sake of time and clarity your talk should focus on the 2-dimensional case. In particular, emphasise the role played by the "positive boundary subcategory" (denoted $\mathcal{Y}_b \subset \mathcal{Y}$ in [MT01]) and of "homological stability" (due to Harer in the 2-dimensional case and proved in [GRW17] in higher dimensions).

Homological stability is the topic of later talks, and should in this talk be treated as a black box. In light of the available time, you should probably not attempt to say much about the proof of the group-completion theorem.

If you have some time left, you could say a bit about the higher-dimensional case, perhaps explained by analogy to the oriented 2-dimensional case. The torus $S^1 \times S^1$ plays a special role in dimension 2, and in dimension $2n$ a similar role is played by $S^n \times S^n$, see for example the definitions and statements in [GRW17, Section 1.1]. The role of orientations on 2-manifolds is replaced by a tangential structure $\theta : B \to BO(2n)$. The "positive boundary subcategory" from dimension 2 is replaced by a subcategory $\mathcal{C}_\theta^{n-1} \subset \mathcal{C}_\theta$ whose cobordisms are $(n-1)$-connected relative to their outgoing boundary, cf. [GRW17, Section 7]. (In the notation of op.cit. you can take $L = \emptyset$ for the purposes of this talk.)

Talk 5. Talks 1–3 introduced the cobordism category $\mathcal{C}_{2n}$ and the tangentially-structured version $\mathcal{C}_\theta$, and related $\Omega B\mathcal{C}_{2n}$ to $\Omega^\infty MTO(2n)$ and similarly $\Omega B\mathcal{C}_\theta \simeq \Omega^\infty MT\theta$, while talk 4 related the stable homology of diffeomorphism groups to $\Omega B\mathcal{C}_\theta^{n-1}$, where $\mathcal{C}_\theta^{n-1} \subset \mathcal{C}_\theta$ is the subcategory whose morphisms are the cobordisms that are $(n-1)$-connected relative to their outgoing boundary. The purpose of this talk and talk 6 is to understand the inclusion functor $\mathcal{C}_\theta^{n-1} \subset \mathcal{C}_\theta$, and in particular understand circumstances under which it induces an equivalence of group completions. This is the topic of [GRW14, Section 3].

This talk should first outline the strategy: define the filtration $\mathcal{C}_\theta = \mathcal{C}_\theta^{-1} \subset \mathcal{C}_\theta^0 \subset \cdots \subset \mathcal{C}_\theta^{n-1}$ and state that we will study one step at a time and prove that $\Omega B\mathcal{C}_\theta^{\kappa-1} \xrightarrow{\simeq} \Omega B\mathcal{C}_\theta^\kappa$ is an equivalence for all $\kappa < n$.

Explain the statement of [GRW14, Theorem 3.4], which can be paraphrased as having a contractible choice of *surgery data*. Taking that result for granted, explain how [GRW14, Theorem 3.1] is proven, in as much detail as you have time for. In the notation of op.cit., make the simplifying assumptions $N = \infty$ and $L = \emptyset$ and omit both from the notation. You should strongly consider focusing only on the case $d = 2$ and $\kappa = 0$, where it is easier to draw pictures of what's going on (in particular, draw a picture of a morphism that is not in $\mathcal{C}_2^0$, and how it is modified by a surgery move to be in $\mathcal{C}_2^0$ as in [GRW14, Figures 2 and 3]). You can also safely ignore the tangential structures θ, which do not play an important role in this step.

Talk 6. This talk should sketch some of the ingredients in [GRW14, Theorem 3.4]. An important ingredient is the simplicial technique in [GRW14, Section 6.2], which may be read independently of the rest of the paper. Try to present this as a self-contained tool in the homotopy theory of simplicial spaces.

Then explain how it is applied in [GRW14, Section 6.3].

2. Homological stability

This series of talks aims to explain the simplest example of a homological stability theorem for diffeomorphism groups, namely in the case of $B\mathrm{Diff}(W_g^{2n}, D^{2n})$ with $W_g = \#^g S^n \times S^n$. These talks follow the unpublished preprint [GRW12], which proves the result of [GRW18a] in this special case, making various simplifications.

Talk 7. This talk should present the proof of homological stability of $B\mathrm{Diff}(W_g^{2n}, D^{2n})$ taking Corollary 4.9 for granted. More precisely, it should define the moduli spaces $\mathscr{M}_g$ following Definition 1.1, give a definition of the semi-simplicial spaces $\overline{K}_\bullet(W_{g,1})$ taking what is needed from Defintion 4.1 (and the paragraph before Corollary 4.9), *state* Corollary 4.9, and then explain Section 5 in as much detail as possible. Doing so will involve using Corollaries 4.4 and 4.5 too, which (with a small variation of their proofs) can be deduced from the statement of Corollary 4.9.

Talk 8. This talk should explain the reduction from geometry to algebra, i.e. the deduction of Corollary 4.9 from (Charney's) Theorem 3.2. You will need to use two technical results from Section 2, Theorem 2.4 and Corollary 2.8, but will not have time to prove them: instead you should state them carefully and indicate their plausibility.

The main steps are: (i) to define $((-1)^n, \Lambda_n)$-quadratic modules and in particular the one associated to $W_{g,1}$, and hence define the map

$$K^\delta(W_{g,1}) \longrightarrow K^a(\pi_n(W_{g,1}), \lambda, \alpha);$$

(ii) to deduce Lemma 4.2 from Theorem 3.2, which uses the general simplicial method described in Theorem 2.4; (iii) to deduce Theorem 4.6 (and hence Corollary 4.9) by comparing the discrete and topologised semi-simplicial spaces, using Corollary 2.8. (Some discussion of discrete versus topologised may have appeared in Talk 6.)

Talk 9. This talk should give an overview of the proof of Theorem 3.2, the high-connectivity of the algebraic complex $|K^a(M)|$. Rather than following Charney's paper [Cha87] it is perhaps simpler to present the argument given in Section 4 of [GRW18a], which proves a slightly more general result than Charney's theorem in the special case that the ring in question is $\mathbb{Z}$, which is what is needed here.

3. The general formulation

Talk 10. This talk should explain the *statements* of Corollaries 1.8 and 1.9 of [GRW17], which are the natural results which follow from appropriately generalising the methods discussed in the talks so far. You may wish to consult Section 4 of [GRW18b] for another perspective.

You should then present the extended example of hypersurfaces $V_d \subset \mathbb{CP}^4$ described in Section 5.3 of [GRW18b] in detail (spend *at least* half the talk on this).

4. Finiteness properties of diffeomorphism groups [after Kupers, Weiss]

This series of talks aims to explain one of the main results of [Kup17], namely that the homology of $B\mathrm{Diff}_\partial(D^{2n})$ has finite type as long as $2n \geq 6$. The general strategy follows an idea of Weiss [Wei15] which allows for the use of embedding calculus in the study of diffeomorphism groups.

Talk 11. This talk should explain the theory of Embedding Calculus, following [Wei99] and [BdBW13]. Particular care should be given to describe the layers in the embedding calculus tower, especially in the case of manifolds with boundary. Convergence of the tower under the codimension ≥ 3 assumption should simply be stated, with an explanation of what "codimension" means in this context. You should collaborate with the speaker for Talk 14 to make sure that you provide all the tools that will be necessary in that talk.

Talk 12. This talk should construct the Weiss fibration

$$B\mathrm{Diff}_\partial(D^d) \longrightarrow B\mathrm{Diff}_\partial(M) \longrightarrow B\mathrm{Emb}^{\cong}_{1/2\partial}(M, M)$$

and explain the proof that it deloops, following Section 4 of [Kup17].

Talk 13. This talk should present Kreck's analysis of the mapping class group $\Gamma_{g,1} = \pi_0(\mathrm{Diff}_\partial(W_{g,1}))$. This is Proposition 3 of [Kre79] in the special case $M = W_g$ (you will need to mention Cerf's "concordance implies isotopy" theorem to get rid of the $\tilde{\ }$'s). Kreck's argument has several external references, and you should do your best to summarise them.

Talk 14. This talk should combine the ingredients of the last three talks to prove the finiteness of $B\mathrm{Diff}_\partial(D^{2n})$ for $2n \geq 6$, i.e. Theorem 5.1 of [Kup17]. This may look short, but involves proving Theorem 3.2 for $M = W_{g,1}$ which in turn involves proving Proposition 3.11 and Proposition 3.15 for these manifolds. Proposition 3.11 concerns finiteness of the mapping class group of $W_{g,1}$, and should be deduced from Talk 13 (and the theorem of Borel–Serre that arithmetic groups are virtually of type (F)) rather than the more general results used in [Kup17]. Proposition 3.15 concerns finiteness of the identity component of $\mathrm{Emb}^{\cong}_{1/2\partial}(W_{g,1}, W_{g,1})$, and should be done following the given proof.

5. Positive scalar curvature [after Botvinnik–Ebert–Randal-Williams]

This series of talks aims to give an idea of the application by Botvinnik–Ebert–Randal-Williams [BERW17] of cobordism category methods to understanding spaces $\mathcal{R}^+(M)$ of Riemannian metrics of positive scalar curvature of a smooth manifold M.

Talk 15. You should start by defining the spaces $\mathcal{R}^+(W)_{h_0,h_1}$ of positive scalar curvature metrics on a cobordism W, and discuss briefly the construction of Hitchin's secondary index invariant (this workshop is not on index-theory, so you will have to simply assert several things). You then carefully formulate the result to be discussed in these talks, namely Theorem A.

You should spend the rest of the talk explaining some of the basic constructions in Section 2.2, and the Cobordism Theorem (Theorem 2.3.1) and its basic application (Spin-cobordism invariance of the space of psc metrics). Finally you should explain the statement of Theorem 3.6.1 and how it reduces Theorem A to the case $d = 6$ following Remark 1.2.6.

Talk 16. This talk should start by discussing the Abelianness Theorem (Theorem 4.1.2), and how it follows from the Cobordism Theorem (Theorem 2.3.1) by a formal argument of Eckmann–Hilton flavour. It should then outline the proof of Theorem B in the 6-dimensional case. The focus should be on constructing the map $\Omega^{\infty+1}MTSpin(6) \to \mathcal{R}^+(D^6)_{h_\circ^5}$, and the index-theoretic part of the argument should be sketched only briefly. Finally, you should say something about Theorem 5.2.1, that

$$MTSpin(6) \longrightarrow \Sigma^{-6}MSpin \longrightarrow \Sigma^{-6}ko$$

is surjective on rational homotopy (you should *not* present this in detail in the talk: it is a routine piece of algebraic topology).

6. Classifying spaces for fibrations [after Berglund–Madsen]

This series of talks aims to give an overview of the work of Berglund–Madsen [BM13, BM14] concerning the classifying spaces $B\mathrm{hAut}_\partial(W_{g,1})$ for fibrations with fibre $W_{g,1}$. (For reasons of time we shall skip the applications to $B\widetilde{\mathrm{Diff}}(M)$, the classifying space for "block diffeomorphisms", in the Berglund–Madsen papers.)

Despite the apparent conceptual similarities between $B\mathrm{Diff}_\partial(W_{g,1})$ and $B\mathrm{hAut}_\partial(W_{g,1})$—the former space classifies smooth fiber bundles while the latter classifies fibrations, in both cases with fibres $W_{g,1}$ and trivialised boundary—the methods of study are quite different, and the calculations produce rather dissimilar-looking answers.

Talk 17. This talk will concern the abstract rational homotopy theory in [BM14, Sections 2–3] and the references therein. Start by explaining what it means for a map between simply connected spaces to be a rational equivalence, and discuss the connection (due to Quillen) to differential graded Lie algebras (DGLAs).

Then discuss mapping spaces in rational homotopy theory, focusing on how to extract a DGLA model for $B\mathrm{hAut}(X)_{\mathbb{Q}}$ given a DGLA model for $X_{\mathbb{Q}}$. The space $B\mathrm{hAut}(X)$ is likely not simply connected even when X is, so in practice one must separately study the discrete group $\pi_0(\mathrm{hAut}(X))$ and the rational homotopy type of the classifying space of the identity component of $\mathrm{hAut}(X)$. The latter is modeled by Lie algebras of derivations, while $\pi_0(\mathrm{hAut}(X))$ in practice often has to be studied "by hand".

Talk 18. This talk will discuss the rational homotopy theory of the spaces $B\mathrm{hAut}(W_{g,1})$, where $W_{g,1} = D^{2n}\#g(S^n \times S^n)$. Specialise the abstract results from Talk 17 to this space, and discuss how the Lie algebras of derivations compare to graph complexes (i.e. first half of [BM14, Theorem 9.1]).

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