Talbot 2019 Talk 9

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The goal of this talk is to prove Charney's theorem, and thus finally completing the proof of homological stability.

Recall that the notion of a quadratic module depends on two parameters, $\varepsilon = \pm 1$ and $\Lambda \subseteq \mathbb{Z}$ a subgroup. The only combinations relevant are

$$(\varepsilon, \Lambda) \in \{(1, \{0\}),\ (-1, 2\mathbb{Z}),\ (-1, \mathbb{Z})\}.$$

A quadratic module is then a triple (M, λ, α) where M is a $\mathbb{Z}$-module, $\lambda : M \otimes M \to \mathbb{Z}$ is an ε-bilinear form, and $\alpha : M \to \mathbb{Z}/\Lambda$ is a function satisfying

1. $\alpha(ax) = a^2\alpha(x)$ for all $a \in \mathbb{Z}$
2. $\alpha(x + y) = \alpha(x) + \alpha(y) + \lambda(x, y)$

The standard example of a quadratic module is the hyperbolic module

$$\mathcal{H} = \mathbb{Z}\{e\} \oplus \mathbb{Z}\{f\}, \quad \lambda = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}, \quad \alpha(e) = \alpha(f) = 0.$$

Definition. Let M bee a quadratic module. The *Witt index* $g(M)$ is

$$g(M) = \sup\{g \in \mathbb{N} \mid \exists \mathcal{H}^{\oplus g} \to M\}.$$

It is clear that $g(M \oplus \mathcal{H}) \geq g(M) + 1$, but we would like this to be exactly equal. Therefore, we define

Definition. The *stable Witt index* $\bar{g}(M)$ is

$$\bar{g}(M) = \sup\{g(M \oplus \mathcal{H}^{\oplus k}) - k \mid k \geq 0\}.$$

Example. We have $g(\mathcal{H}^{\oplus k}) = k$, since it is obviously at least k, and cannot be larger than k for dimension reasons. Therefore $\bar{g}(\mathcal{H}^{\oplus k}) = k$ as well.

Recall that if M is a quadratic module, then $K^a(M)$ is the simplicial complexes with vertices given by morphisms $h : \mathcal{H} \to M$, and $\{h_0, \dots h_p\}$ for a p-simplex if the images of the h_i are orthogonal with respect to λ.

Theorem (Charney)**.** *Let $g \in \mathbb{N}$ and M a quadratic module with $\bar{g}(M) \geq g$. Then $K^a(M)$ is $\lfloor \frac{g-4}{2} \rfloor$-connected and locally weakly Cohen–Macaulay of dimension $\lfloor \frac{g-1}{2} \rfloor$.*

Definition. A simplicial complex X is *locally weakly Cohen–Macaulay* of dimension n if for every p-simplex σ, the link of σ is $(n-p-2)$-connected.

Thus, a weakly Cohen–Macaulay complex to be a locally weakly Cohen–Macaulay complex that is also $(n-1)$-connected.

Proof. We proceed by induction on g. The base cases are $g = 1, 2, 3, 4, 5$, which we will do at the end.

We first show the locally weakly Cohen–Macaulay condition. Let $\sigma = \{h_0, \ldots, h_p\}$ be a p-simplex, and let M' be the orthogonal complement of the images of the h_i's. Then we have

$$M = M' \oplus \mathcal{H}^{\oplus p+1},$$

and $\bar{g}(M') = \bar{g}(M) - p - 1$. Then observe that the $\mathrm{Lk}(\sigma) = K^a(M')$, so by induction, it is $\lfloor \frac{g-p-1-4}{2} \rfloor \geq (\lfloor \frac{g-1}{2} \rfloor - p - 2)$-connected.

We next turn to the connectivity result. We suppose the result holds up to $g-1$. Given an M, we know in particular that $K^a(M)$ is non-empty, so we can pick an $h : \mathcal{H} \to M$, and let $M' = h(\mathcal{H})^{\perp}$. Then again

$$M = M' \oplus \mathcal{H}.$$

Our strategy is to show that $K^a(M') \to K^a(M)$ is both $n = \lfloor \frac{g-4}{2} \rfloor$-connected and null-homotopic. The null-homotopic part is easy — $K^a(M')$ is the link of the vertex h, so the inclusion factors through the (closure of the) star of h, which is deformation retracts onto h.

To show the map is n-connected, we factor it into two parts note that $h(e)^{\perp} = M' \oplus \mathbb{Z}\{e\}$. So our map factors as

$$K^a(M') \xrightarrow{(1)} K^a(M' \oplus \mathbb{Z}\{e\}) \xrightarrow{(2)} K^a(M).$$

We will show that (1) and (2) are both n-connected. This is based on the following combinatorial fact:

Proposition ([GRW18, Proposition 2.5])**.** *Let X be a simplicial complex and $Y \subseteq X$ a subcomplex. Let $n \in \mathbb{Z}$ be such that for all p-simplices σ in X having no vertex in Y, the complex $Y \cap \mathrm{Lk}(\sigma)$ is $(n-p-1)$-connected. Then $Y \hookrightarrow X$ is n-connected.*

(1) There is a projection map $M' \oplus \mathbb{Z}\{e\} \to M'$ which induces a retraction $\pi : K^a(M' \oplus \mathbb{Z}\{e\}) \to K^a(M')$. Let σ be a simplex in $K^a(M' \oplus \mathbb{Z}\{e\})$. The geometric observation is that

$$K^a(M') \cap \mathrm{Lk}_{K^a(M' \oplus \mathbb{Z}\{e\})}(\sigma) = \mathrm{Lk}_{K^a(M')}(\pi(\sigma)).$$

Since $\bar{g}(M') = \bar{g}(M) - 1$, we know that this is $(\lfloor \frac{g-2}{2} \rfloor - p - 2)$-connected by induction, as desired. So the connectivity follows from the proposition.

(2) This is similar to the previous one. Let $\sigma = \{h_0, \dots, h_p\}$ be a p-simplex in $K^a(M)$, and let M'' be the orthogonal complement of the images of the h_i, so that

$$M = M'' \oplus \mathcal{H}^{\oplus p+1}.$$

We also have

$$K^a(M' \oplus \mathbb{Z}\{e\}) \cap \mathrm{Lk}_{K^a(M)}(\sigma) = K^a(M'' \cap (M' \oplus \mathbb{Z}\{e\})).$$

Since $\bar{g}(M'' \cap (M' \oplus \mathbb{Z}\{e\})) \geq g - p - 2$, we are done by induction.

We are left with the base cases. Suppose $\bar{g}(M) \geq 4$. We want to show path-connectedness of $K^a(M)$. In fact, let us make the stronger assumption that $g(M) \geq 4$.

By assumption, we can pick an $h_0 : \mathcal{H} \to M$ such that $g(h_0(\mathcal{H})^\perp) \geq 3$. We shall find a path from h_0 to any other vertex $h_1 : \mathcal{H} \to M$. In fact, we shall find a path of length 2. That is, we want a morphism $h : \mathcal{H} \to h_0(\mathcal{H})^\perp \cap h_1(\mathcal{H})^\perp$. Equivalently, by definition, we want to show that

$$g(h_0(\mathcal{H})^\perp \cap h_1(\mathcal{H})^\perp) \geq 1.$$

To do so, we use the algebraic fact that if M is a quadratic module and $\ell : M \to \mathbb{Z}^a$ is linear, then $g(\ker \ell) \geq g(M) - a$. Once we have this, we simply observe that $h_0(\mathcal{H})^\perp \cap h_1(\mathcal{H})^\perp$ is the kernel of the composite

$$h_0(\mathcal{H})^\perp \hookrightarrow M \twoheadrightarrow h_1(\mathcal{H}).$$

This proves the case where $g(M) \geq 4$, and we now show that it is also true when $\bar{g}(M) \geq 4$. In fact, we will show that $\bar{g}(M) \geq 4$ implies $g(M) \geq 4$.

We note that the proof of our geometric cancellation theorem also proves the following:

Cite the theorem in talk 7

Proposition. *Let M, N be quadratic modules such that $M \oplus \mathcal{H} \cong N \oplus \mathcal{H}$. If $K^a(M \oplus \mathcal{H})$ is connected, then $M \cong N$.*

Equipped with this, suppose M is such that $\bar{g}(M) \geq 4$. Then there exists a quadratic module N with $g(N) \geq 4$ and a large k such that $M \oplus \mathcal{H}^{\oplus k} \cong N \oplus \mathcal{H}^{\oplus k}$, and $g(N \oplus \mathcal{H}^{\oplus k}) \geq k + 4$. Thus $K^a(N \oplus \mathcal{H}^{\oplus k})$ is connected, and by induction, we see that $M \cong N$.

Finally, we have to show that $K^a(M) \neq \emptyset$ when $\bar{g}(M) \geq 2$. The same argument shows that there is some N with $g(N) \geq 2$ and $M \oplus \mathcal{H} \cong N \oplus \mathcal{H}$. Then $g(N \oplus \mathcal{H}) \geq 3$ and M is the kernel of a map $N \oplus \mathcal{H} \to \mathcal{H}$, hence $g(M) \geq 1$. □

References

[GRW18] Søren Galatius and Oscar Randal-Williams. Homological stability for moduli spaces of high dimensional man- ifolds i. *Journal of the American Mathematical Society*, 31(1):215, 2018.