Talbot 2019 Talk 8

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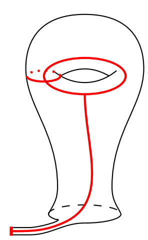
In the previous talk, we proved homological stability using the high connectivity of a certain semi-simplicial space, which we did not prove. In this talk, we will make progress towards the proof by reducing it to the connectivity of an algebraic analogue of this complex, which will finally be proved in the next talk.

Recall we defined $\overline{K_p(W_{g,1})}$ to be the space of embeddings $\phi_0, \ldots, \phi_p : H \hookrightarrow W_{g,1}$ satisfying certain conditions, one of which is that the images have to be disjoint. We will write $\overline{K(W_{g,1})}$ for $|\overline{K_\bullet(W_{g,1})}|$, and same for the other versions we introduce.

We define another semi-simplicial space $K_p(W_{g,1})$ with a slightly less restrictive condition. Contained in H is a *core* $C \cong S^n \vee S^n \vee [0,1] \cup D^{2n-1} \times \{0\}$ that H deformation retracts onto.



$K_p(W_{g,1})$ is then defined in the same way as $\overline{K_p(W_{g,1})}$, except we only require the images of the cores to be disjoint, not the entire image. It is easy to see that the inclusion $\overline{K_p(W_{g,1})} \to K_p(W_{g,1})$ is a weak homotopy equivalence, hence so is $\overline{K(W_{g,1})} \to K(W_{g,1})$.

Finally, we define $K_p^\delta(W_{g,1})$ to be the same semi-simplicial space, but is given the discrete topology. This can equivalently be thought of as a simplicial complex, and we will freely confuse the two. This is a natural thing to consider, because we want to compare to an algebraic analogue of this, where the p-simplices do not come with a topology.

The algebraic analogue of this complex captures the behaviour of the intersection form plus some quadratic data. For any $n \in \mathbb{N}$, there is a fiber sequence

$$S^n \to BO(n) \to BO(n+1),$$

which gives rise to a long exact sequence

$$\pi_{n+1} B\mathrm{O}(n+1) \xrightarrow{\partial} \pi_n S^n \xrightarrow{i} \pi_n B\mathrm{O}(n) \longrightarrow \pi_n B\mathrm{O}(n+1) \to 0$$

The map $i : \pi_n S^n \cong \mathbb{Z} \to \pi_n B\mathrm{O}(n)$ sends the generator to the map classifying the tangent bundle of S^n. So by the Hopf invariant one problem, we know

$$\Lambda_n \equiv \operatorname{im} \partial = \ker i = \begin{cases} 0 & n \in 2\mathbb{Z} \\ \mathbb{Z} & n = 1, 3, 7 \\ 2\mathbb{Z} & \text{otherwise} \end{cases}$$

We then have

$$\operatorname{im} i = \mathbb{Z}/\Lambda_n = \ker(\pi_n B\mathrm{O}(n) \to \pi_n B\mathrm{O}(n+1) \cong \pi_n B\mathrm{O}).$$

We make the following definition:

Definition. A *quadratic module* is a triple (M, λ, α), where M is a $\mathbb{Z}$-module, $\lambda : M \otimes M \to \mathbb{Z}$ is an $(-1)^n$-symmetric bilinear form, and $\alpha : M \to \mathbb{Z}/\Lambda_n$ is a function such that

1. $\alpha(ax) = a^2 \alpha(x)$ for all $a \in \mathbb{Z}$
2. $\alpha(x + y) = \alpha(x) + \alpha(y) + \lambda(x, y)$

Note that since α takes values in $\mathbb{Z}/\Lambda_n$ but λ takes values in $\mathbb{Z}$, the second condition does not imply that λ is completely determined by α.

The most basic example of a quadratic module is the hyperbolic module:

Example. The hyperbolic module is defined by

$$\mathcal{H} = \left(\mathbb{Z}\{e\} \oplus \mathbb{Z}\{f\}, \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}, \alpha(e) = \alpha(f) = 0 \right).$$

We will later see that this is the quadratic module associated to $S^n \times S^n$.

Observe that if M is any quadratic module, then any morphism $\mathcal{H} \to M$ is automatically injective since it has to preserve the bilinear form.

Definition. Let M be a quadratic module. We define $K^a(M)$ to be the simplicial complex whose vertices are morphisms $v : \mathcal{H} \to M$ and p-simplicies are sets $\{v_0, \ldots, v_p\}$, where the v_i are orthogonal with respect to λ.

The algebraic connectivity theorem is due to Charney:

Theorem (Charney)**.** *$K^a(\mathcal{H}^{\oplus g})$ is $\lfloor \frac{g-5}{2} \rfloor$-connected.*

In the remainder of the talk, we will show that that this theorem implies the geometric version. More precisely, we will use this theorem to show that $K^\delta(W_{g,1})$ is $\lfloor \frac{g-5}{2} \rfloor$-connected, and then use this to deduce that $K(W_{g,1})$ is $\lfloor \frac{g-5}{2} \rfloor$-connected.

First we need to describe how we can get a quadratic module from a manifold. Let W^{2n} be a stably parallelizable $(n-1)$-connected manifold. Then by Hurewicz, we have $\pi_n(W) \cong H_n(W; \mathbb{Z})$ and the intersection form on H_n gives us a pairing $\lambda : \pi_n(W) \otimes \pi_n(W) \to \mathbb{Z}$.

To obtain the map α, we use Haefliger's theorem:

Theorem (Haefliger [Hae62])**.** *Any $x \in \pi_n(W)$ can be represented by an embedded sphere if $n \geq 3$, and is unique up to isotopy if $n \geq 4$.*

If $x \in \pi_n(W)$, then after picking a representative by an embedded sphere, the normal bundle gives an element $\alpha(x) \in \pi_n B\mathrm{O}(n)$, and since it is stable trivial, it is in fact in $\mathbb{Z}/\Lambda_n$. For $n \geq 4$, this is well-defined since the representative is unique, and when $n = 3$, α takes values in the zero group, hence is necessarily well-defined. One can then check that $(\pi_n(W), \lambda, \alpha)$ gives a quadratic module.

Example. The quadratic module of $W_{g,1}$ is $\mathcal{H}^{\oplus g}$.

Lemma. *If $K^a(\mathcal{H}^{\oplus g})$ is $\lfloor \frac{g-5}{2} \rfloor$-connected, then so is $K^\delta(W_{g,1})$.*

Proof. The above procedure gives us a map of simplicial complexes $K^\delta(W_{g,1}) \to K^a(\mathcal{H}^{\oplus g})$. To show that $\pi_k(K^\delta(W_{g,1})) = 0$, we need to show that any map $\partial I^{k+1} \to |K^\delta(W_{g,1})|$ extends to a map on I^{k+1}. By assumption, we know the extension exists when we compose down to $|K^a(\mathcal{H}^{\oplus g})|$. So we have to solve the lifting problem

$$\begin{array}{ccc} \partial I^{k+1} & \longrightarrow & |K^\delta(W_{g,1})| \\ \downarrow & \nearrow & \downarrow \\ I^{k+1} & \xrightarrow{g} & |K^a(\mathcal{H}^{\oplus g})|. \end{array}$$

Our strategy is to pick a simplicial approximation of triangulation of g, and then lift simplex by simplex. It turns out we can pick a particularly nice triangulation, using the fact that $K^a(\mathcal{H}^{\oplus g})$ is *weakly Cohen–Macaulay.*

Definition. A simplicial complex X is *k-weakly Cohen–Macaualay* if it is $(k-1)$-connected and the link[1] of any p-simplex is $(k-p-2)$-connected.

We will use the following fact without proof:

Lemma. *If X is k-weakly Cohen–Macaulay, and $f : \partial I^k \to |X|$ is simplicial with respect to some triangulation of ∂I^k, then there is an extension $I^k \to |X|$ such that g is simplex-wise injective on the interior of I^k, i.e. it doesn't collapse simplices to lower-dimensional simplices.*

It is not too difficult to see that $K^a(\mathcal{H}^{\oplus g})$ is $\lfloor \frac{g-3}{2} \rfloor$-weakly Cohen–Macaulay, since if $\sigma = \{v_0, \ldots, v_p\}$ is a p-simplex, then the link is $K^a(\mathrm{im}(v_0, \ldots, v_p)^\perp)$. If $g - p - 1 < 4$, then the connectivity we desire is < -1, hence is automatic. Otherwise, $\mathrm{im}(v_0, \ldots, v_p)^\perp \cong \mathcal{H}^{\oplus g-p-1}$ by cancellation (c.f. the proof in the geometric case in the previous talk). So we are done by induction.

We now return to our lifting problem. After picking the simplicial approximation of g as above, we need to decide how to lift the vertices in I^{k+1}. We put a total order on the set of vertices and do the lifting one by one. Let v be a vertex in the interior of I^{k+1}, and consider the embedding $g(v) : \mathcal{H} \hookrightarrow \pi_n(W_{g,1})$.

[1] The *star* of a simplex σ is the union of all simplices that contain σ (which is an open subset of the simplicial complex). The *link* is then obtained by taking the closure of the star and then removing the star.

Then $g(v)(e)$ and $g(v)(f)$ can be represented by embedded spheres whose algebraic intersection with each other is $+1$, while the algebraic intersection with other neighbours that have already been lifted is 0. Applying Whitney's trick, we can assume the previous sentence is true with "algebraic" replaced by "geometric" (and that the intersection is transverse). We can then thicken $g(v)(e)$ and $g(v)(f)$ to an embedding of $\mathcal{H}$ into $W_{g,1}$ of the desired form, and use this as our lift. One then checks that this works. $\square$

Lemma. *If $K^\delta(W_{g,1})$ is $\lfloor \frac{g-5}{2} \rfloor$-connected, then so is $K(W_{g,1})$.*

Proof sketch. Consider the bisimplicial space $D_{p,q} = K_{p+q+1}(W_{g,1})$, topologized as a subspace of $K_p(W_{g,1}) \times K_q^\delta(W_{g,1})$.

By construction, this comes with augmentations $D_{p,q} \to K_p(W_{g,1})$ and $D_{p,q} \to K_p^\delta(W_{g,1})$. The first observation is that the following diagram is commutative up to homotopy

$$\begin{array}{ccc} |D_{\bullet,\bullet}| & \longrightarrow & |K_\bullet^\delta(W_{g,1})| \\ & \searrow & \downarrow \\ & & |K_\bullet(W_{g,1})| \end{array}$$

It suffices to show that $|D_{\bullet,\bullet}| \to |K_\bullet(W_{g,1})|$ is $\lfloor \frac{g-5}{2} \rfloor$-connected. Then since it factors through a $\lfloor \frac{g-5}{2} \rfloor$-connected space, the target must be $\lfloor \frac{g-5}{2} \rfloor$-connected.

To show that $|D_{\bullet,\bullet}| \to |K_\bullet(W_{g,1})|$ is $\lfloor \frac{g-5}{2} \rfloor$-connected, it suffices to show that $|D_{p,\bullet}| \to K_p(W_{g,1})$ is $(\lfloor \frac{g-5}{2} \rfloor - p)$-connected. In fact, we will show it is $\lfloor \frac{g-p-4}{2} \rfloor$-connected.

The idea is that the map $|D_{p,\bullet}| \to K_p(W_{g,1})$ is like a Serre fibration, so the connectivity of this map is given by the connectivity of the literal fiber (after a degree shift). It is not a Serre fibration, but it is what is known as a Serre *micro*fibration, and it turns out this *also* means the connectivity of the map is given by the connectivity of the literal fiber.

To understand the literal fiber, let $z = ((t_i, \phi_i)) \in K_p(W_{g,1})$, and W the complement of the $\phi_i(C)$'s. Then the fiber over z admits a map to $K^a(\pi_n(W)) \cong K^a(\mathcal{H}^{\oplus g+p-1}$, and since the target is $\lfloor \frac{g-p-6}{2} \rfloor$-connected, the argument in the previous lemma shows that the fiber over z is also $\lfloor \frac{g-p-6}{2} \rfloor$-connected. $\square$

References

[Hae62] Andrefliger. Plongements diffntiables de varis dans varis. *Commentarii mathematici Helvetici*, 36:47–82, 1961/62.