Talbot 2019 Talk 7

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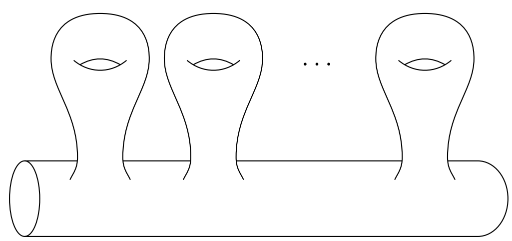
The goal of the coming three talks is to understand and prove (a specific instance of) homological stability. The main theorem is about the $B\mathrm{Diff}$ of some manifolds, so we begin by describing the manifolds we are interested in:

$$W_{g,1} = \#_g S^n \times S^n \setminus D^{2n}.$$

We elect to draw this as follows:



Since $W_{g,1}$ has a boundary, we are interested in a slight variation of $B\mathrm{Diff}$. We let $\mathrm{Diff}^\partial(W_{g,1})$ be the group of self-diffeomorphisms of $W_{g,1}$ that fix a neighbourhood of the boundary, and consider $B\mathrm{Diff}^\partial(W_{g,1})$.

The obvious inclusion $W_{g,1} \hookrightarrow W_{g+1,1}$ induces a map $B\mathrm{Diff}^\partial(W_{g,1}) \to B\mathrm{Diff}^\partial(W_{g+1,1})$, and the main theorem is

Theorem. *The map*

$$H^k(B\mathrm{Diff}^\partial(W_{g,1}); \mathbb{Z}) \to H^k(B\mathrm{Diff}^\partial(W_{g+1,1}); \mathbb{Z})$$

is an isomorphism when

$$k \leq \frac{g-4}{2}.$$

Our approach to proving this theorem involves having an explicit geometric model for $B\mathrm{Diff}^\partial(W_{g,1})$. In the case of a manifold M without boundary, an explicit model for $B\mathrm{Diff}(M)$ is given by the space of submanifolds of $\mathbb{R}^\infty$ that are diffeomorphic to M. More precisely,

$$B\mathrm{Diff}(M) = \operatorname*{colim}_{N\to\infty} \mathrm{Emb}(M, \mathbb{R}^N)/\,\mathrm{Diff}(M).$$

In the case of manifolds with boundary, we need to say a few more words. We let $\mathbb{H}^N = [0,\infty)\times\mathbb{R}^{N-1}$, and we consider embeddings of $W_{g,1}$ into $\mathbb{H}^N$ with some prescribed boundary behaviour.

First of all, we fix a diffeomorphism between a neighbourhood of the boundary with $S^{2n-1}\times[0,1)$. We then fix an embedding $S^{2n-1}\hookrightarrow\mathbb{R}^{N-1}$, which extends to an embedding of $S^{2n-1}\times[0,1)\hookrightarrow\mathbb{H}^N$. We do this for all sufficiently large N, in a way that is compatible with the inclusions $\mathbb{H}^N\hookrightarrow\mathbb{H}^{N+1}$.

We then let $\mathrm{Emb}^\partial(W_{g,1},\mathbb{H}^N)$ be the space of embeddings of $W_{g,1}$ into $\mathbb{H}^N$ that agrees with the above embedding on $S^{2n-1}\times[0,\varepsilon)$ for some ε. We then have

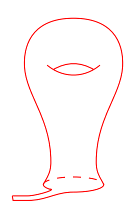
$$B\mathrm{Diff}^\partial(W_{g,1}) = \operatorname*{colim}_{N\to\infty} \mathrm{Emb}(W_{g,1},\mathbb{H}^N)/\,\mathrm{Diff}^\partial(W_{g,1}).$$

Heuristically, this is the space of submanifolds of $\mathbb{H}^\infty$ that are diffeomorphic to $W_{g,1}$ and is standard near the boundary. For this reason, we will also denote this space by $\mathcal{M}_g$.

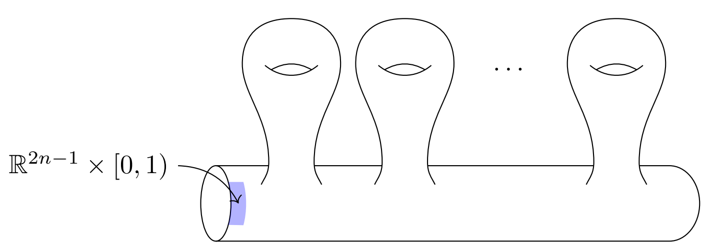
To prove the theorem, we need a way to measure the difference between $B\mathrm{Diff}^\partial(W_{g,1})$ and $B\mathrm{Diff}^\partial(W_{g+1,1})$, and the technical devices we employ is due to Quillen (unpublished).

The proof involves a semi-simplicial space $\overline{K_\bullet(W_{g,1})}$, which we proceed to define. The reader must for now believe that this space is in fact important.

Defining this semi-simplicial space requires a bit of preparation. Define H to be $W_{1,1}$, but pictured in a different way.[1]



We fix an embedding of $D^{2n-1}\times[0,1)$ into H, which we picture as the end of the tail. We also fix an embedding of $\mathbb{R}^{2n-1}\times[0,1)$ into $W_{g,1}$:



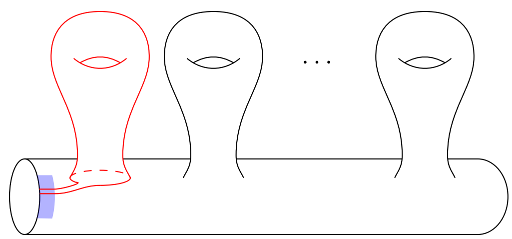
[1] This is called G in the workshop, because an algebraic analogue of this will also be called H. Here we shall abuse the luxury of multiple typefaces instead

With all this preparation, we can now define the semi-simplicial complex $K_\bullet(W_{g,1})$.

Definition. $\overline{K_0(W_{g,1})}$ is the space of all embeddings $\phi : H \hookrightarrow W_{g,1}$ such that under the above diffeomorphisms, ϕ takes the following form near the boundary:

$$\phi(x, s) = (x + te_1, s),$$

where $e_1 \in \mathbb{R}^{2n-1}$ is the basis vector in the 1 direction, t is some fixed constant (which is, of course, uniquely determined by ϕ), and this is required to hold on some $D^{2n-1} \times [0, \varepsilon) \subseteq H$.



We define $\overline{K_p(W_{g,1})}$ to be the subset of $\overline{K_0(W_{g,1})^{p+1}}$ consisting of $(\phi_0, \ldots \phi_p)$ such that the images of the embeddings are disjoint, and the associated t_i satisfy $t_0 < \cdots < t_p$.

Note that the latter condition only serves to fix an ordering on the vertices. This is necessary since we are working with semi-simplicial spaces, which demand an ordering on vertices of a simplex.

The main fact we will use about this semi-simplicial space is the following theorem:

Theorem. $\left|\overline{K_\bullet(W_{g,1})}\right|$ *is* $\lfloor \frac{g-5}{2} \rfloor$*-connected.*

This theorem will be proved in the next talk, and the goal of the current talk is to see how homological stability follows from this.

What does this theorem mean? If $g \geq 4$, then in particular, this tells us the semi-simplicial complex is path-connected. The connected components of $K_0(W_{g,1})$ are isotopy classes of embeddings $\phi : H \hookrightarrow W_{g,1}$. Given any two vertices $\phi_1, \phi_2 : H \hookrightarrow W_{g,1}$, there is an edge between them iff the embeddings are disjoint. So the fact that $\left|\overline{K_\bullet(W_{g,1})}\right|$ is connected means given any two embeddings, we can find a sequence of embeddings starting from ϕ_1 and ending at ϕ_2, such that each embedding is disjoint from the previous one (or rather, is isotopic to an embedding that is disjoint from the previous one). One can think of higher connectivity as telling us how canonical this sequence is.

This observation allows us to quickly prove a number of corollaries:

Corollary (Transitivity)**.** *If $g \geq 4$ and $\phi_1, \phi_2 \in \overline{K_0(W_{g,1})}$, then there is a diffeomorphism $f : W_{g,1} \to w_{g,1}$ such that $\phi_2 = f \circ \phi_1$ and $f|_{\partial W_{g,1}}$ is isotopic to the identity.*

Proof. We claim this is obvious if they are disjoint. An explicit such diffeomorphism is described in the original papers, which we shall not reproduce [GRW12, Corollary 4.4]. The proof then follows from the previous observation. $\square$

Corollary (Cancellation). *If $\phi \in \overline{K_0(W_{g,1})}$, then $W_{g,1} \setminus \phi(H)$ is diffeomorphic to $W_{g-1,1}$.*

Proof. By the previous corollary, we only have to prove this for a "standard" embedding of H into $W_{g,1}$, for which it is "evident". $\square$

We now proceed to prove our main theorem.

Let X_p be the space of submanifolds of $\mathbb{H}^\infty$ diffeomorphic to $W_{g,1}$, together with $p+1$ disjoint embeddings of H into that submanifold (in a way that is standard at the boundary, as usual). More precisely, we define

$$X_p = \mathrm{Emb}^\partial(W_{g,1}, \mathbb{H}^\infty) \times \overline{K_p(W_{g,1})} / \mathrm{Diff}^\partial(W_{g,1}).$$

Forgetting the second factor leads to a map $X_p \to \mathcal{M}_g$ whose fiber is $\overline{K_p(W_{g,1})}$. This is in fact a fiber sequence, which is the same fact as our previous claim that our explicit model of $B\mathrm{Diff}^\partial(W_{g,1})$ is a legitimate model.

Taking geometric realizations, we have a fiber sequence

$$\begin{array}{ccc} \left|\overline{K_\bullet(W_{g,1})}\right| & \longrightarrow & |X_\bullet| \\ & & \downarrow \\ & & \mathcal{M}_g \end{array}$$

and we know the fiber is $\lfloor \frac{g-5}{2} \rfloor$-connected, and this tells us $H_q(X_\bullet) \cong H_q(\mathcal{M}_g)$ when q is small compared to g.

Now there is a spectral sequence

$$E^1_{p,q} = H_q(X_p) \Rightarrow H_{p+q}(|X_\bullet|),$$

and the d^1 differential is given by

$$\mathrm{d}^1 = \sum (-1)^i (d_i)_*,$$

where the d_i are the face maps.

Claim. $X_p \simeq \mathcal{M}_{g-p-1}$ and the d_i are all (weakly) homotopy to the standard inclusion $\mathcal{M}_{g-p-1} \hookrightarrow \mathcal{M}_{g-p}$.

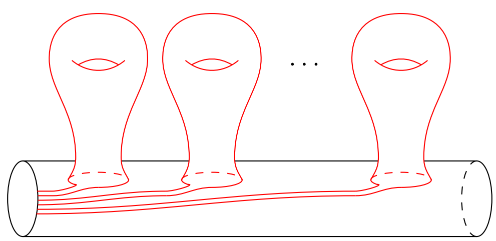
Assuming the claim, the theorem follows immediately. Since all the d_i's are in particular homotopic to each other, the d^1 differentials are either 0 or i_*, depending on the parity of p. So the spectral sequence looks like this:

$$
\begin{array}{ccccccccc}
H_4(\mathcal{M}_{g-1}) & \xleftarrow{0} & H_4(\mathcal{M}_{g-2}) & \xleftarrow{i_*} & H_4(\mathcal{M}_{g-3}) & \xleftarrow{0} & H_4(\mathcal{M}_{g-4}) & \xleftarrow{i_*} & H_4(\mathcal{M}_{g-5}) \\
H_3(\mathcal{M}_{g-1}) & \xleftarrow{0} & H_3(\mathcal{M}_{g-2}) & \xleftarrow{i_*} & H_3(\mathcal{M}_{g-3}) & \xleftarrow{0} & H_3(\mathcal{M}_{g-4}) & \xleftarrow{i_*} & H_3(\mathcal{M}_{g-5}) \\
H_2(\mathcal{M}_{g-1}) & \xleftarrow{0} & H_2(\mathcal{M}_{g-2}) & \xleftarrow{i_*} & H_2(\mathcal{M}_{g-3}) & \xleftarrow{0} & H_2(\mathcal{M}_{g-4}) & \xleftarrow{i_*} & H_2(\mathcal{M}_{g-5}) \\
H_1(\mathcal{M}_{g-1}) & \xleftarrow{0} & H_1(\mathcal{M}_{g-2}) & \xleftarrow{i_*} & H_1(\mathcal{M}_{g-3}) & \xleftarrow{0} & H_1(\mathcal{M}_{g-4}) & \xleftarrow{i_*} & H_1(\mathcal{M}_{g-5}) \\
H_0(\mathcal{M}_{g-1}) & \xleftarrow{0} & H_0(\mathcal{M}_{g-2}) & \xleftarrow{i_*} & H_0(\mathcal{M}_{g-3}) & \xleftarrow{0} & H_0(\mathcal{M}_{g-4}) & \xleftarrow{i_*} & H_0(\mathcal{M}_{g-5})
\end{array}
$$

By induction on g, we know that the i_* appearing in the diagram above are isomorphisms below a line of slope $-\frac{1}{2}$, so in the E^2 page, below this line, the only non-zero entries are $H_q(\mathcal{M}_{g-1})$ in the first column. Since the spectral sequence converges to $H_{p+q}(\mathcal{M}_g)$ in low degrees, the theorem follows after some careful keeping track of indices.

It remains to convince ourselves that our claim is true.

Proof of claim. We first produce the desired map $\mathcal{M}_{g-p-1} \hookrightarrow X_p$, and then show that this is a weak homotopy equivalence. Fix an embedding of $W_{p+1,2} \hookrightarrow [0, p+1] \times \mathbb{R}^\infty$ equipped with $p+1$ embeddings of H.



The map $\mathcal{M}_{g-p-1} \hookrightarrow X_p$ is obtained by taking a submanifold, translating in the first coordinate by $p+1$, and then adding the copy of $W_{p+1,2}$, and the embeddings of H are the ones we have fixed in the $W_{p+1,2}$.

We claim this map is a weak homotopy equivalence. Observe that this map is a homeomorphism onto the subspace of X_p consisting of submanifolds and embeddings such that the intersection with $[0, p+1] \times \mathbb{R}^\infty$ is our standard copy of $W_{p+1,2}$ with the standard $p+1$ embeddings of H.

The first step we take is also the non-obvious step. There is a projection map

$$X_p \to \mathrm{Emb}^\partial \left(\coprod_{p+1} H, \mathbb{H}^\infty \right)$$

that picks out the embeddings of H. The point is that this is a fibration whose target is contractible, so the inclusion of the fiber into X_p is a weak homotopy equivalence.

The fiber consists of submanifolds and embeddings such that the embeddings are "standard". Noticing that the image of the embeddings (plus a small neighbourhood of the boundary component at $\{0\} \times \mathbb{R}^\infty$) is isotopy equivalent to the whole $W_{p+1,2}$, it is not difficult to see that we can deform any compact family of submanifolds (and embeddings) in the fiber to ones contained in the image of $\mathcal{M}_{g-p-1}$, thereby completing the proof of the first part.

Recall that $d_i : X_p \to X_{p-1}$ is obtained by forgetting the i^{th} embedding of H. Therefore, the following diagram commutes essentially by definition:

$$\begin{array}{ccc} \mathcal{M}_{g-p-1} & \xrightarrow{\sim} & X_p \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle d_p} \\ \mathcal{M}_{g-p} & \xrightarrow{\sim} & X_{p-1} \end{array} .$$

One then sees that there is an isotopy from our embedding $W_{p+1,2} \hookrightarrow [0, p+1] \times \mathbb{R}^\infty$ to an embedding with the same image but swaps two adjacent embeddings of H. This then provides a homotopy between the composites

$$\mathcal{M}_{g-p-1} \xrightarrow{\sim} X_p \xrightarrow{d_i} X_{p-1} .$$

This completes the proof. □

References

[GRW12] Søren Galatius and Oscar Randal-Williams. Homological stability for moduli spaces of high dimensional man- ifolds. 2012.