Talk 6: Prepping for surgery

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In the previous talk we had a map $\mathcal{D}^{\kappa}_{\bullet,\bullet} \to \mathcal{D}^{\kappa-1}_{\bullet}$, and a theorem saying that it induces a weak homotopy equivalence after geometric realization. The purpose of this talk is to explore more of the proof of this result.

Let $X = (W, t, \varepsilon) \in \mathcal{D}^{\kappa-1}$. Define a (semi-)simplicial set $Z_\bullet(X)$ by

$$Z_0(X) = \{\text{surgery data } (\Lambda, \delta, e : \Lambda \times V \to \mathbb{R}^\infty) \text{ for } X\},$$
$$Z_p(X) = \{(q+1) - \text{tuples of disjoint surgery data}\},$$

where face operators are given by forgetting entries from the tuples. Rephrasing the definition in the previous talk,

$$\mathcal{D}^{\kappa}_{p,q} = \left\{(x, y) \mid x \in \mathcal{D}^{\kappa-1}_p, y \in Z_q(x)\right\}.$$

Remark. Since pairwise disjoint disjointedness can be detected by the 1-simplices, mush of the information we want is contained in the 1-skeleton.

Definition 1. A simplicial space $X_\bullet$ is said to be *augmented* if we have a space X_{-1} and a map $X_0 \xrightarrow{\varepsilon} X_{-1}$ such that $\varepsilon \circ d_0 = \varepsilon \circ d_1$.

Note that any simplicial space has the trivial augmentation map with $X_{-1} = \{*\}$. Our particular case of interest will be the augmented simplicial space $\mathcal{D}^{\kappa}_{p,\bullet} \to \mathcal{D}^{\kappa-1}_p$.

Definition 2. An *augmented topological flag complex* (ATFC) $X_\bullet \to X_{-1}$ is an augmented simplicial space $X_\bullet$ satisfying

1. $X_n \to X_0 \times_{X_{-1}} \cdots \times_{X_{-1}} X_0$ is a homeomorphism onto its image, which is open.
2. $(v_0, \ldots, v_n) \in X_0 \times_{X_{-1}} \cdots \times_{X_{-1}} X_0$ is in X_n if $(v_i, v_j) \in X_1$ for all $i < j$.

Think about the second condition as a 'triangle-filling condition'. Of course, our augmented simplicial space $\mathcal{D}^{\kappa}_{p,\bullet} \to \mathcal{D}^{\kappa-1}_p$ is an ATFC.

Theorem 1. (6.2 in [GRW14]) *Let $X_\bullet \to X_{-1}$ be an ATFC and suppose that*

i) $\varepsilon : X_0 \to X_{-1}$ has local sections,

ii) ε is surjective,

iii) for all $p \in X_{-1}$ and for all nonempty finite $\{v_1, \ldots, v_n\} \subset \varepsilon^{-1}(p)$, there exists $v \in \varepsilon^{-1}(p)$ such that $(v_i, v) \in X_1$ for all i.

Then $\|X_\bullet\| \to X_{-1}$ is a weak homotopy equivalence.

We want to obtain a level-wise weak equivalence $\|\mathcal{D}^{\kappa}_{p,\bullet}\| \xrightarrow{\sim} \mathcal{D}^{\kappa-1}_p$. Instead of applying the previous result directly, we want to make a slight modification of the simplicial spaces in question. Let $\tilde{Z}_\bullet(X)$ consist of surgery data (Λ, δ, e) with a softened condition: e is only required to be an embedding in a neighborhood of $C = [0, 2] \times D^\kappa \times \{0\} \subset [0, 2] \times \mathbb{R}^\kappa \times \mathbb{R}^{d-\kappa}$. We call C the *core*. There is a corresponding $\tilde{\mathcal{D}}^{\kappa}_{\bullet,\bullet}$ defined using $\tilde{Z}_\bullet$.

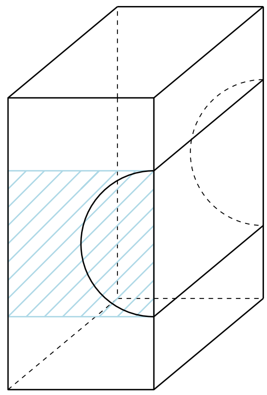

Figure 1: The core $C \subset \overline{V}$ in light blue.

Claim. $\tilde{\mathcal{D}}^{\kappa}_{\bullet,\bullet} \leftarrow \mathcal{D}^{\kappa}_{\bullet,\bullet}$ is a weak homotopy equivalence after geometric realization.

Using the theorem above, the map $\tilde{\mathcal{D}}^{\kappa}_{\bullet,\bullet} \to \mathcal{D}^{\kappa-1}_{\bullet}$ given by $(W,t,\varepsilon),(\Lambda,\zeta,e) \to (W,t,\varepsilon)$ yields a weak equivalence $\|\tilde{\mathcal{D}}^{\kappa}_{p,\bullet}\| \to \mathcal{D}^{\kappa-1}_{p}$. For this we shall need the hypotheses $2\kappa \leq d-1$ and $\kappa+1+d < N$. Let us sketch how the conditions *i), ii)* and *iii)* are verified.

First one would like to check that $\tilde{\mathcal{D}}^{\kappa}_{p,0} \to \mathcal{D}^{\kappa-1}_{p}$ has local sections. The idea here is to consider $(W,t,\varepsilon) \in \mathcal{D}^{\kappa-1}_{p}$ and $s_0 < t_0 - \varepsilon$, $s_1 > t_p + \varepsilon$ regular values for $x_1 : W \to \mathbb{R}$ such that $(x_1 \circ e)(\Lambda \times \overline{V}) \subset (s_0,s_1)$. One takes a neighborhood $U \subset \mathcal{D}^{\kappa-1}_{p}$ where the restriction of each manifold over $[s_0,s_1]$ is diffeomorphic to $M = W|_{[s_0,s_1]}$ and so define a map $F : U \to \mathrm{Emb}_{\partial}(M,[s_0,s_1]\times(-1,1)^N)/\,\mathrm{Diff}(M)$. This map can be lifted to $\tilde{F} : U \to \mathrm{Emb}_{\partial}(M,[s_0,s_1]\times(-1,1)^N)$. To take care of surgery data there is a way to get $\varphi : U \to \mathrm{Diff}([s_0,s_1]\times(-1,1)^N)$ (possibly reducing U) such that $\tilde{F}(u)$ is obtained from $W|_{[s_0,s_1]}$ by applying φ. A local section will be then given by

$$u = (W,t,\varepsilon) \mapsto (\tilde{F}(u),t,\varepsilon),(\Lambda,\delta,\varphi(u)\circ e).$$

The second condition to check is the surjectivity of $\tilde{\mathcal{D}}^{\kappa}_{p,0} \to \mathcal{D}^{\kappa-1}_{p}$. Fix $(W,t,\varepsilon) \in \mathcal{D}^{\kappa-1}_{p}$. We need to show that $\tilde{Z}_0(W,t,\varepsilon)$ is non-empty. To find the discs where surgery is to be performed, let X be a finite relative CW-complex having only cells of dimension $\geq \kappa$ such that $(X, W|_{t_i}) \xrightarrow{\sim} (W|_{[t_{i-1},t_i]}, W|_{t_i})$ is a weak equivalence. Here the right hand side is $(\kappa-1)$-connected. We can take the κ-cells and consider the diagram

$$\begin{array}{ccc} (D^{\kappa},\partial D^{\kappa}) & \longrightarrow & (X, W|_{t_i}) \\ & \searrow & \downarrow \\ & & (W|_{[t_{i-1},t_i]}, W|_{t_i}). \end{array}$$

By dimension considerations, each map $(D^{\kappa},\partial D^{\kappa}) \to (W|_{[t_{i-1},t_i]}, W|_{t_i})$ can be deformed to be an embedding. We need to be able to extend the embedding of $\partial_{-}D^{\kappa+1}$ such that what lives over zero is sent 'all the way to the right'. This can be done because W is $(\kappa-1)$-connected relative to the outgoing boundary. By the assumptions on dimension, we can use a similar general position argument to show that this embedding can be extended in a 'nice' way so that it defines an embedding in a neighborhood of $[-2,0]\times\mathbb{R}^{\kappa}\times\{0\}$, and extend once again to the whole $\overline{V}$ in a way that it satisfies the defining conditions of surgery data. The surjectivity of $\mathcal{D}^{\kappa}_{p,0} \to \mathcal{D}^{\kappa-1}_{p}$ follows.

To conclude we see how condition iii) holds. Fix $X = (W,t,\varepsilon)$ and $v_1,\ldots,v_k \in \tilde{Z}_0(X)$. We need to find $v \in \tilde{Z}_0(X)$ such that v_i and v are disjoint. The idea is to take initially $v = v_1$ and deform the embeddings to force trasversality with the cores of the v_i's. Once again, under our assumptions on the dimension, this transversality is just the disjointness we are looking for.

References

[GRW14] Søren Galatius and Oscar Randal-Williams. Stable moduli spaces of high-dimensional manifolds. *Acta Math.*, 212(2):257–377, 2014.