Talk 5

Speaker: Jan Steinebrunner

This talk follows the ideas in [GRW14]. Let $\mathcal{C}$ be the cobordism category as defined in previous talks.

Definition 1. We define the subcategory $\mathcal{C}^\kappa \subset \mathcal{C}$ having morphisms (W, t) with $W \subset [0, t]$ such that (W, W_t) is κ-connected.

Example

- The category $\mathcal{C}^{-1}$ is no other than $\mathcal{C}$ itself.
- The category $\mathcal{C}^0$ is the subcategory of $\mathcal{C}$ for which every component of every morphism is required to have non-empty boundary. This was denoted by D in previous talks.

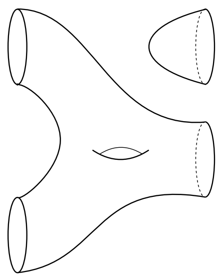

Figure 1: A morphism in $\mathcal{C}^0$

These subcategories define a filtration

$$\mathcal{C} = \mathcal{C}^{-1} \supset \mathcal{C}^0 \supset \mathcal{C}^1 \supset \cdots .$$

Theorem 1. *Suppose that $2\kappa \leq d-2$ and $k+1+d \leq N$. Then the inclusion induces a weak homotopy equivalence*

$$\|\mathcal{C}^\kappa(\mathbb{R}^N)\| \xrightarrow{\sim} \|\mathcal{C}^{\kappa-1}(\mathbb{R}^N)\|.$$

From now on, we will assume $N = \infty$ for simplicity. A version with tangential structures is true also, but we will not be concerned with this in the current talk.

To get a first idea of the proof of the theorem, let us sketch how to go from $\mathcal{C}^{-1} = \mathcal{C}$ to $\mathcal{C}^0$:

1. Take some morphism in $\mathcal{C}^{-1}$, say (W, t). The components of W may or may not have outgoing boundary.
2. Choose some points and corresponding disjoint paths in W. Each path goes from the chosen point to some point in $\{t\} \times \mathbb{R}^\infty$. The collection of chosen points must have at least one element from each component without outgoing boundary, in order to guarantee the desired connectedness. As a fact, the space of such choices is contractible. That is the content of the next lecture.

3. Do surgery by pulling the bordism along the paths for $r \in [0,1]$, obtaining a parameterized family W^r such that $W^0 = W$ and $W^1 \in \mathcal{C}^0$. As a warning note, the manifold W^r constructed this way will not be an allowed morphism in $\mathcal{C}^{-1}$ for $0 < r < 1$. We shall see later how to circumvent this difficulty.

4. Do this consistently to obtain

$$\begin{tikzcd}
\|\mathcal{C}^{-1}\| \times \{0\} \arrow[d] \arrow[dr, "\mathrm{id}"] & \\
\|\mathcal{C}^{-1}\| \times [0,1] \arrow[r] & \|\mathcal{C}^{-1}\| \times \{0\} \\
\|\mathcal{C}^{-1}\| \times \{1\} \arrow[u] \arrow[r] & \|\mathcal{C}^{0}\|, \arrow[u]
\end{tikzcd}$$

which tells us that $\|\mathcal{C}^0\| \subset \|\mathcal{C}^{-1}\|$ is a deformation retract.

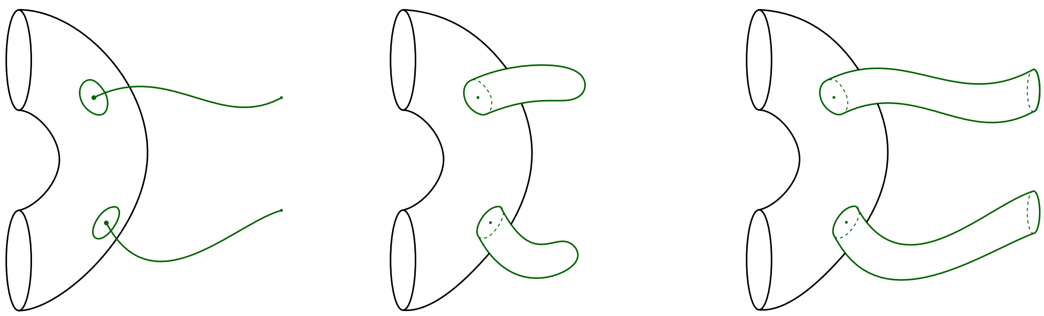

Figure 2: Going from $\mathcal{C}^{-1}$ to $\mathcal{C}^0$

Definition 2. The category $\mathcal{D}^\kappa$ is the poset with elements (t, ε, W) with $t \in \mathbb{R}$, $\varepsilon > 0$, $W \in \Psi(\mathbb{R} \times (-1,1)^\infty)$, and such that $(t-\varepsilon, t+\varepsilon)$ consists of regular values of $x_1 : W \to \mathbb{R}$. The ordering is given by $(a, \varepsilon, V) < (a, \varepsilon', W)$ if and only if $V = W$, $a + \varepsilon < b - \varepsilon'$ and $(W|_{[a,b]}, W_b)$ is κ-connected. We also define $\mathcal{D}_p^\kappa := N_p \mathcal{D}^\kappa$.

Example The left hand side in the figure below illustrates an element of $\mathcal{D}_2^{-1}$ which is not in $\mathcal{D}_2^0$. That is because the morphism lying over $[t_1, t_2]$ has a component with no outgoing boundary. If we moved t_1 to the right as in the right hand side, it would now belong to $\mathcal{D}_2^0$.

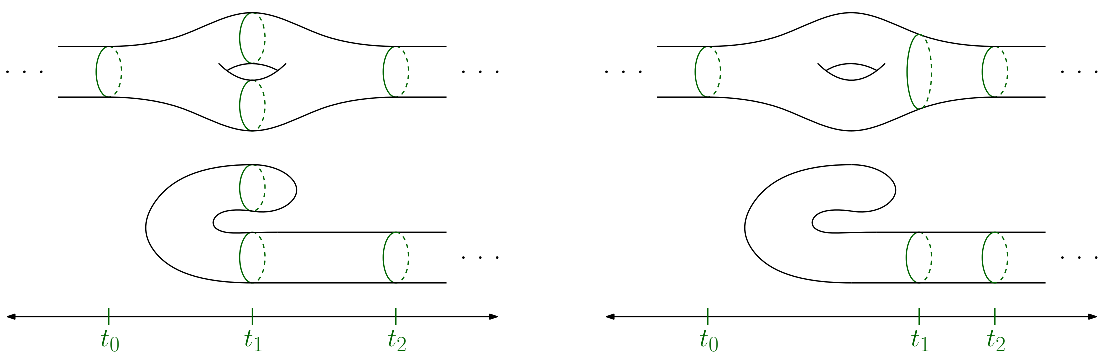

Figure 3: An element of $\mathcal{D}_2^{-1}$ and an element of $\mathcal{D}_2^0$.

Let $V = (-2,0) \times \mathbb{R}^\kappa \times \mathbb{R}^{d-\kappa}$ and denote $h : \overline{V} \to [-2,0]$ the projection onto the first factor. Also let $P_0 = \partial_- D^{\kappa+1} \times \mathbb{R}^{d-\kappa}$, where ∂_- is the lower hemisphere of the boundary of the unit disk. We want to use these pictures to define the surgery data: It consists of an embedding e of V and $e(P_0)$ is the intersection of a given bordism (i.e. a 1-simplex in the nerve of D^{-1}) with the image of V.

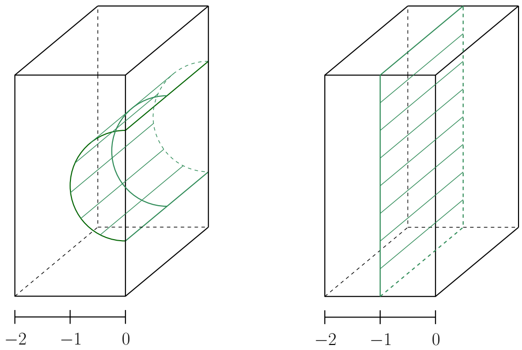

Figure 4: $P_0 \subset \overline{V}$ for $\kappa = 1$, $d = 1$ (on the left) and $\kappa = 0$, $d = 2$ (on the right).

Definition 3. Let $x = (t, \varepsilon, W) \in \mathcal{D}_p^{\kappa-1}$. Then a *surgery datum for* x is Λ a finite set together with $e : \Lambda \times V \hookrightarrow \mathbb{R} \times \mathbb{R}^\infty$ and $\delta : \Lambda \to \{0, \dots, p+1\}$, such that for each λ

i) on $e_\lambda^{-1}((t_i - \varepsilon_i, t_i + \varepsilon_i))$, $x_1 \circ e_\lambda$ is an affine rescaling of h,

ii) $h^{-1}(-2)$ is sent to $(-\infty, t_0 - \varepsilon_0)$ by $x_1 \circ e_\lambda$,

iii) $h^{-1}(-3/2)$ is sent to $(-\infty, t_i - \varepsilon_i)$ by $x_1 \circ e_\lambda$, for $i = \delta(\lambda)$,

iv) $h^{-1}(0)$ is sent to $(t_p + \varepsilon_p, \infty)$ by $x_1 \circ e_\lambda$,

v) $e_\lambda^{-1}(W) = P_0$,

vi) and for all $i \in \{1, \dots, p+1\}$, $\left(W|_{[t_{i-1},t_i]}, W|_{t_i} \cup \bigcup_{\lambda \in \delta^{-1}(i)} e_\lambda(P_0)\right)$ is κ-connected.

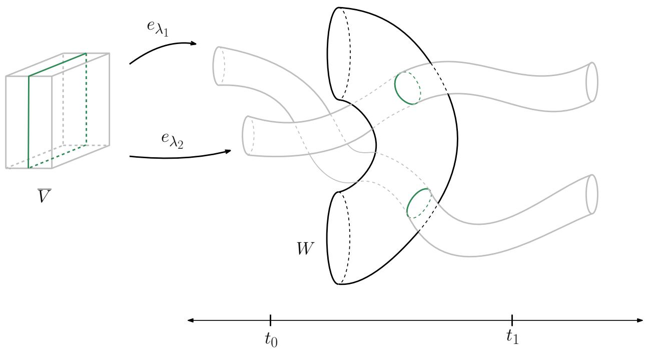

Figure 5: A surgery datum for certain W in $\mathcal{D}_1^{-1}$.

Performing a surgery

The idea now is to use a "standard family" P_t for $t \in [0,1]$ as illustrated in Figure 6 below. Given $(t, \varepsilon, W) \in \mathcal{D}_p^{\kappa-1}$ and surgery data $(\Lambda, e : \Lambda \times V \hookrightarrow \mathbb{R} \times \mathbb{R}^\infty)$, define

$$K_e^t(W) = W \setminus e(P_0 \times \Lambda) \cup e(P_t \times \Lambda).$$

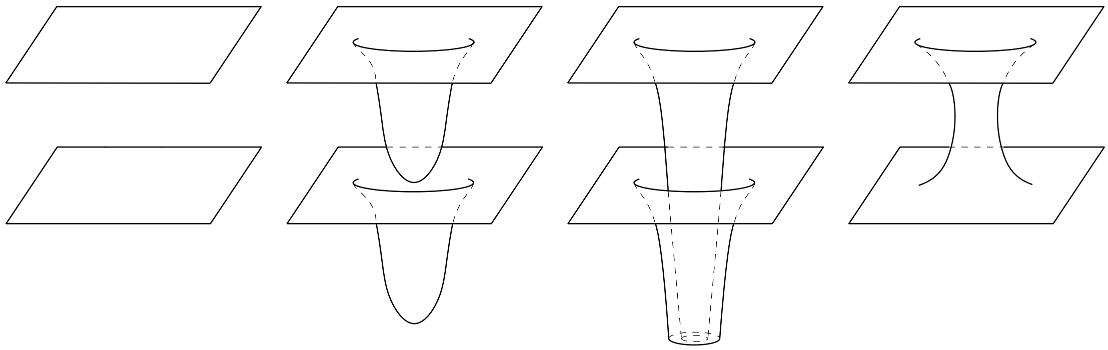

Figure 6: The standard family P_t.

As t progresses from 0 to 1, $K_e^t(W)$ corresponds to the process illustrated in Figure 2. There is a caveat at this point, however. Even though we start with an element of $\mathcal{D}_p^{\kappa-1}$ for $t = 0$, there is no guarantee that $K_e^t(W)$ will stay in $\mathcal{D}_p^{\kappa-1}$ for intermediate times. This motivates the following definition.

Definition 4. Let $X_\bullet^\kappa$ be the (semi-)simplicial space such that X_r^κ consists of triples $(t, \varepsilon.W)$ as in $\mathcal{D}_\bullet^\kappa$ but now isolated singular values are allowed over $(t-\varepsilon, t+\varepsilon)$. Moreover, there is the extra condition that if $b < c$ are regular values in $(t-\varepsilon, t+\varepsilon)$ then $(W|_{[b,c]}, W_c)$ is κ-connected.

According to the following lemma, enlarging our category like this does not change much for our purposes. As a remark, the proof makes use of Sard's theorem.

Lemma 1. *The inclusion* $\mathcal{D}_\bullet^\kappa \hookrightarrow X_\bullet^\kappa$ *induces an equivalence* $\|\mathcal{D}_\bullet^\kappa\| \simeq \|X_\bullet^\kappa\|$.

Now we proceed to define a bisimplicial space that will be our main technical artifact. It can be thought of as a fibration over $\mathcal{D}^{\kappa-1}$, where the fiber over an element consists of all possible choices of surgery data for such element.

Definition 5. Define a bisimiplicial space $\mathcal{D}_{\bullet,\bullet}^\kappa$

$$\mathcal{D}_{p,q}^\kappa = \left\{(x,y) \mid x \in \mathcal{D}_p^{\kappa-1}, y = (e^0, \dots, e^q) \text{ and } e^i \text{ are pairwise disjoint surgery data for } x\right\}.$$

The following result formalizes the idea that these choices of surgery data form a contractible space. The proof is the main topic of the next talk.

Lemma 2. *There is an equivalence* $\|\mathcal{D}_{\bullet,\bullet}^\kappa\| \simeq \|\mathcal{D}_\bullet^{\kappa-1}\|$.

To conclude the proof of Theorem 1, define a homotopy

$$\zeta : [0,1] \times \mathcal{D}_{p,q}^\kappa \times \Delta^q \to X_p^{\kappa-1}$$
$$(t, ((a,\varepsilon,W), (e^0, \dots, e^q)), (s_0, \dots, s_q)) \mapsto (a, \varepsilon, K_{e^q}^{t\bar{s}_q} \circ K_{e^{q-1}}^{t\bar{s}_{q-1}} \circ \cdots \circ K_{e^0}^{t\bar{s}_0}(W))$$

where $\bar{s}_i = \frac{s_i}{\max s_j}$. We get a diagram

$$\begin{tikzcd}
\|\mathcal{D}_\bullet^\kappa\| \arrow[r, "\sim"] \arrow[d] & \|X_\bullet^\kappa\| \arrow[d] & \|\mathcal{C}^\kappa\| \arrow[l, "\sim"'] \arrow[d] \\
\|\mathcal{D}_{\bullet,\bullet}^\kappa\| \arrow[r, "\zeta(0,-)"] \arrow[ur, "\zeta(1,-)"] & \|X_\bullet^{\kappa-1}\| & \|\mathcal{C}^{\kappa-1}\|, \arrow[l, "\sim"']
\end{tikzcd}$$

where $\zeta(0,-)$ is the composite of the equivalences in the two previous lemmas. Thus the vertical arrows are also weak equivalences as desired.

References

[GRW14] Søren Galatius and Oscar Randal-Williams. Stable moduli spaces of high-dimensional manifolds. *Acta Math.*, 212(2):257–377, 2014.