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Why $\Omega^\infty MTSO(2)$ can be understood as a group completion

REVIEW Recall $\mathcal{C}$ topological category, $N_\bullet\mathcal{C}$ nerve, $\|\mathcal{C}\| = \|N_\bullet\mathcal{C}\|$ geometric realization of the nerve.

We have an equivalence $\Omega\|\mathbf{Cob}_2^{or}\| \xrightarrow{\sim} \Omega^\infty MT^\theta(2) = \Omega^\infty MTSO(2)$. We had an auxiliary category D and we constructed a homology equivalence $\Omega\|D\| \overset{H_*\text{equiv.}}{=}$ certain group completion

4.1 Group completion and localization

Group completion is functor

$$\{\text{discrete monoids}\} \to \mathbf{Grp}$$
$$M \mapsto MM^{-1}$$

with a map $M \to MM^{-1}$ such that for every map $M \to G$ to a group G, there exists a unique map of groups $MM^{-1} \to G$ making the diagram commute. We have a bijection $\mathrm{Hom}_{\mathbf{Grp}}(MM^{-1}, G) \simeq \mathrm{Hom}_{\mathbf{Mon}}(M, G)$

Example.
1. $M = (\mathbb{N}, +)$ then $MM^{-1} = \mathbb{Z}$
2. $M = (Vect_{\mathbb{C}}, \oplus)$ then $MM^{-1} = \mathbb{Z}$ given by rank
3. Proj $(\mathbb{C}[x]), \oplus$ $\mathbb{Z}$
4. $(\mathbb{C}, \times)$ becomes 0
5. $Mat_m(\mathbb{C})$ becomes 0

Definition. Suppose M and E_1 space/topological monoid. Then the group completion o f M is determined up to homotopy by

$$\mathrm{Hom}_{E_1\text{-spaces}}(MM^{-1}, G) \simeq \mathrm{Hom}_{E_1\text{-spaces}}(M, G)$$

where G= a group = an E_1-space with inverse up to higher homotopies.

Definition. M topological monoid, then $BM = B\|\mathbf{M}\|$ where $\mathbf{M}$ is M regarded as a one-object topological category.

Recall that we have a map $M \to \Omega BM$ given by $(BM)_1 = S^1 \wedge M \to BM$ which gives a map due to the the adjunction.

Lemma. *$M \to \Omega BM$ is an equivalence iff $\pi_0(M)$ is a group.*

Theorem. *$M \to \Omega BM$ is the group completion of M*

Proof. This follows from the following facts

- ΩBM is a group.
- $\Omega B \circ U$ where U is the forgetful functor $\mathbf{Grp} \to \mathbf{Mon}$ is the identity.

□

Theorem. *(Group completion theorem, v.1) Recall that we have a $\pi_0 M$ action on the homology $H_*(M)$ and let $\pi_0 M$ be finitely generated. Suppose that if we invert multiplication by $\pi_0(M)^{-1}$ on right, $\pi_0(M)$ is invertible on left.*

$$H_*(M)[\pi_0(M)^{-1}] \xrightarrow{\sim} H_*(\Omega BM)$$

Pick generators $m_1, \dots, m_k \in \pi_0 M$ which by abuse of notation we consider as elements of M. Let $M_m = \operatorname{colim}\left(M \xrightarrow{\times m} M \xrightarrow{\times m} \cdots\right)$. Let $\left(\left((M_{m_1})_{m_2}\right)\cdots\right)_{m_k} = M_\infty$.

Remark. $H_*(M_\infty) = H_*(M)[\pi_0(M)^{-1}]$ because homology commutes with *nice* colimits.

If M is disrete, then $M_\infty = MM^{-1}$.

Example (picture of group completion of $\mathbb{N}$ as identification space of a lattice).

We can make $(-)_\infty$ into a functor, which gives a map $M_\infty \to (\Omega BM)_\infty = \Omega BM$

Reformulation $M_\infty \to \Omega BM$ is a homology equivalence.

Proof.

$$\begin{array}{ccc} M_\infty = \pi^{-1}(*) & \longrightarrow & EM \times_M M_\infty \\ & & \downarrow{\scriptstyle \pi} \\ & & EM \times_M \{*\} \end{array}$$

where upper right is contractible. I claim that $\pi^{-1}(pt) \to \operatorname{hofib} \pi$ is a homology equivalence. □

Example.
- $M = \coprod_{n\in\mathbb{N}} B\,\mathrm{Gl}_n(\mathbb{C})$ and $b \in \mathrm{Gl}_1(\mathbb{C})$. Then multiplication by b gives a shift map and $M_\infty = \coprod_{n\in\mathbb{Z}} \operatorname{colim}_k B\,\mathrm{Gl}_k(\mathbb{C})$ where the colimit is BU.

$$\begin{array}{ccc} \Omega\|\mathbf{Cob}_2^{or}\| & \longrightarrow & \Omega^\infty MTSO(2) \\ \uparrow & & \\ \Omega\|D\| & \longrightarrow & \mathbb{Z} \times B\,\mathrm{Diff}(\Sigma_{\infty,2}, \partial) \end{array}$$

where the lower arrow is a homology equivalence. What is D?

Recall $\mathrm{Obj}(\mathbf{Cob}_2^{or}) = \mathrm{Emb}(\coprod S^1, \mathbb{R}^\infty)/\,\mathrm{Diff}^+\left(\coprod S^1\right)$. $\mathrm{Mor}(\mathbf{Cob}_2^{or}) = \mathrm{Emb}(F, \mathbb{R}^\infty \times [0,t])/\,\mathrm{Diff}(F, \partial F)$ where we require elements of of $\mathrm{Diff}(F, \partial F)$ to fix a (collar) neighborhood of the boundary:

$$\begin{array}{ccc} \mathrm{Diff}^{technical}(F) & \longrightarrow & \mathrm{Diff}(F) \\ \downarrow & & \downarrow \\ \mathrm{Diff}(\partial_{in} F \times [0,\varepsilon)) \times \mathrm{Diff}(\partial_{out} F \times (t-\varepsilon, t]) & \longrightarrow & \mathrm{Diff}(\partial_{in} F) \times \mathrm{Diff}(\partial_{out} F) \end{array}$$

Definition. D is the largest subcategory of $\mathcal{C}$ such that every connected component of the bordism has nonempty outgoing boundary

Theorem. $D\left(\left[\coprod_n S^1 \to \mathbb{R}^\infty\right], \left[S^1 \to \mathbb{R}^\infty\right]\right) = \coprod_{g\geq 0} B\,\mathrm{Diff}\left(\Sigma_{g,n+1}, \partial\right)$

Proof. ??? □

We have a subcategory of the endomorphisms of $S^1 \to \mathbb{R}^\infty$ we have $\mathrm{Mor}\,\mathrm{End}_D(S^1 \to \mathbb{R}^\infty) = \coprod_{g\geq 0} B\,\mathrm{Diff}(\Sigma_{g,2})$ where the (fat) geometric realization of this category is given by $\Omega B\left(\coprod_{g\geq 0} B\,\mathrm{Diff}(\Sigma_{g,2})\right)$

Interlude by Oscar on group completion for categories

Our goal is to say something about $B\,\mathrm{Diff}$ of surfaces. In particular, we want to to say something about connection between topology of morphisms in a category and the topology of ΩB of said category.

Have $\mathcal{C}(X,Y) \to \Omega_{X,Y} B\mathcal{C}$. If $X = Y$, then LHS is a monoid and RHS is a group, so we need to 'do some violence' to make things work out. Turns out the right thing to do is to take/consider homology.

Fix $Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} \cdots$ (X like a variable). Then

$$\operatorname*{colim}_{i\to\infty} H_*(\mathcal{C}(X,Y_i)) \to \operatorname*{colim}_{i} H_*(\Omega_{X,Y_i} B\mathcal{C})$$

RHS as functor of X sends all morphisms to isomorphisms. Also all maps in the colimit (concatenation of paths) are isomorphisms. Want this to be iso–minimum hypothesis says that the LHS sends all morphisms to isomorphisms.

GCT for categories If LHS (as functor of X) sends all morphisms $X' \to X$ to isos, then comparison map is an iso, for all X.

Let $\mathcal{C} = \mathcal{C}_2^{or}$, $Y_i = S^1$, $f_i = S^1 \times S^1 \backslash (D^2 \cup D^2)$ does *not* satisfy the hypothesis: $\operatorname{colim}_i H_*(\mathcal{C}_2^{or}(-,Y_i))$ fails to send morphisms to injections. **Solution** Let D be the subcategory where all components of all cobordisms have outgoing boundary. $D(X,Y_i) \simeq \coprod_{g\geq 0} B\,\mathrm{Diff}(\Sigma_{g,X}) \implies \operatorname{colim}_i D(X,Y_i) \simeq \mathbb{Z} \times B\,\mathrm{Diff}(\Sigma_{\infty,X})$

Homology stab $\implies$ the hypothesis holds for this situation.

Now we just need to show D and $\mathcal{C}$ have the same classifying space.