3 The missing link

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Recall we had $\mathcal{C}_d^n \simeq \psi_d(n,1) \simeq \Omega^{n-1}\psi_d(n,n) \simeq \Omega^{n-1}\operatorname{Th}(\gamma_d^\perp \to BO(d))$ where $\psi_d(n,n) = \Psi_d(\mathbb{R}^n)$. This talk will focus on the second equivalence. Recall that $\psi_d(n,k)$ are defined by the condition that $M \subseteq \mathbb{R}^k \times (-1,1)^{n-k}$.

Theorem. *For $k \geq 2$, there exists a weak homotopy equivalence $\psi_d(n,k-1) \to \Omega\psi_d(n,k)$ (*).*

Let's define this map

$$\begin{aligned} \mathbb{R} \times \psi_d(n,k-1) &\to \psi_d(n,k) \\ (t,M) &\mapsto M - t\cdot e_k \end{aligned}$$

Choose as a basepoint the empty manifold. Then the above map extends to a map $S^1 \wedge \psi_d(n,k-1) \to \psi_d(n,k)$

Definition. For $A \subset \mathbb{R}$ open, define $\psi_d^A(n,k) \subset \psi_d(n,k)$ consist of the manifolds M in $\psi_d(n,k)$ such that $M \cap x_k^{-1}(\mathbb{R}\setminus A) = \varnothing$.

$\psi_d^A(n,k)'$ ... $[-1,1] \cap M \cap x_k^{-1}(\mathbb{R}\setminus A) = \varnothing$

$\psi_d^\circ(n,k)$ ... where $M \cap [-1,1]^n \cap x_k^{-1}(t) = \varnothing$ for some $t \in (0,1)$

e.g. $\psi_d(n,k-1) = \psi_d^{(-1,1)}(n,k)$

$N_\ell\psi_d(n,k-n) \subseteq \mathbb{R}^{\ell+1} \times \psi_d(n,k)$ is the set of pairs (t,M) such that $0 < t_0 \leq \cdots \leq t_\ell < 1$ and $M \in \psi_d^{\mathbb{R}\setminus\{t_0,\ldots,t_\ell\}}(n,k)$. Then we get a simplicial space $N_\bullet\psi_d(n,k-1)$. Then (*) factors through

$$\psi_d(n,k-1) \xrightarrow{(i)} \Omega|N_\bullet\psi_d(n,k-1)| \xrightarrow{(ii)} \Omega\psi_d^\circ(n,k) \xrightarrow{\Omega(incl.),(iii)} \Omega\psi_d(n,k)$$

where the second map is given by $(t,M) \mapsto M$.

Geometric interpretation of the first map: we have $\psi_d(n,k-1) \simeq N_1\psi_d(n,k-1) \to \Omega|N_\bullet\psi_d(n,k-1)|$: empty submanifold is the basepoint.

Goal: show these maps (i), (ii), (iii) are homotopy equivalences.

For (i), need two lemmas

Lemma. *Suppose $X_k \simeq X_1 \times \cdots \times X_1$ (k factors) induced by the degeneracy maps and $\pi_0 X_1$ a group. Then $X_1 \simeq \Omega|X_\bullet|$*[4].

Lemma. *Suppose $A_1,\ldots,A_m \subseteq \mathbb{R}$ disjoint open subsets. Then $\prod \psi_d^{A_i}(n,k) \xrightarrow{\simeq} \psi_d^{\bigcup A_i}(n,k)$.*

Stretching property: $\psi_d^{(a_0,a_1)}(n,k) \simeq \psi_d(n,k-1)$

$\psi_d^{(-\infty,a)}(n,k) \simeq \psi_d^{(a,\infty)}(n,k) \simeq *$

$\pi_0 N_1\psi_d(n,k-1) = \pi_0\psi_d(n,k-1)$ *is a group.*

Why is the last statement true? Becuase $B\mathcal{C}_d^n \simeq \psi_d(n,1)$, then $\pi_0\psi_d(n,1) = \pi_0 B\mathcal{C}_1^n = \mathrm{Obj}/\simeq$

Let's move on to (ii) $|N_\bullet\psi_d(n,k-1)| \to \psi_d^\circ(n,k)$ given by $(t,M) \mapsto M$. Replace the domain by $\psi(-,-)'$ with the prime ('). The fiber over $M \in \psi_d^\circ(n,k)$ is $\{t \in (0,1) \mid M \cap [-1,1]^n \cap x_k^{-1}(t) = \varnothing\} \simeq *$ (note: insufficient that the fiber is contractible, need to do a similar argument to end of last talk)

[4] strictly speaking we get paths that start and end in X_0, but $X_0 \simeq *$ by assumption

Case: (iii) $\psi_d^\circ(n,k) \hookrightarrow \psi_d^\varnothing(n,k)$ is a weak homotopy equivalence. (rhs is the connected component of the basepoint the empty manifold)

Baby case: [picture of a surface with genus 1, unbounded in $\pm\infty$ directions, $a \in (0,1)$. 'push' manifold off to infinity in one direction so that the surface doesn't live over a]

In general: $f : (D^m, \partial D^m) \to (\psi_d^\varnothing(n,k), \psi_d^\circ(n,k))$. Let $a \in \mathbb{R}$. $U_a := \{y \in D^m \mid x_1 : f(y) \to \mathbb{R}$ has no critical points in $\{a\} \times I^{k-1} \times \mathbb{R}^{n-k}\}$ is an open set. By the stretch move, $f \simeq f_1$, where $x_1 : f_1(y) \to \mathbb{R}$ has no critical points in $\{a\} \times \mathbb{R}^{n-1}$. Choose a finite subcover by contractible sets $V_1, \ldots, V_r \subset D^m$ with different regular values. For $y \in V_i$, let $\lambda_i(y)$ be the slice over a_i. Since $\pi_0\psi_d(n,k) = \pi_0\psi_{d-1}(n-1,k-1)$. $[0,1] \times V_i \xrightarrow{\Lambda_i} \psi_{d-1}(n-1,k-1)$ starting at λ_i and going to $\varnothing$.