2 Spaces of not-necessarily compact embedded manifolds

Speaker: Kevin

2.1 Introduction

Recall that a tangential structure is the RHS below

$$\begin{array}{ccccc} E & \longrightarrow & \theta^*\gamma & \longrightarrow & \gamma_d \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & X & \xrightarrow{\theta} & BO(d) = \mathrm{Gr}_d(\mathbb{R}^\infty) \end{array}$$

and θ-structure on a vector bundle $E \to M$ is a bundle map above.

Write $\mathcal{C}_{d(,\theta)}(\mathbb{R}^n)$ be the category with objects $M^{d-1} \subset \mathbb{R}^n$ (and a θ-structure on $TM \oplus \varepsilon^1$) and morphisms cobordisms W (with a θ-structure)

Theorem. *There is a weak-homotopy equivalence*

$$B\mathcal{C}_d \xrightarrow{\sim} \Omega^{\infty-1} MTO(d)$$
$$B\mathcal{C}_{d,\theta} \xrightarrow{\sim} \Omega^{\infty-1} MT\theta(d)$$

The sketch we will follow is a modification of the original theorem [GRW10]. We will define $\Psi_{d,\theta}(\mathbb{R}^n), \psi_{d,\theta}(n,1)$ and show that

Theorem (3.22 of [GRW10])**.** $\Psi_{d,\theta}(\mathbb{R}^n) \simeq \mathrm{Th}(\theta_n^* \gamma_{d,n}^\perp)$ *where*

$$\begin{array}{ccc} X(\mathbb{R}^n) & \longrightarrow & X \\ \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \theta} \\ \mathrm{Gr}_d(\mathbb{R}^n) & \longrightarrow & BO(d) = \mathrm{Gr}_d(\mathbb{R}^n) \end{array}$$

Theorem (3.9 and 3.10)**.** $B\mathcal{C}_{d,\theta}(\mathbb{R}^n) \xrightarrow{\sim} \psi_{d,\theta}(n,1)$

To prove the main result, we need $\psi_{d,\theta}(n,1) \simeq \Omega^{n-1}\Psi_{d,\theta}(\mathbb{R}^n)$ which will be discussed in the next talk.

2.2 Spaces of submanifolds

Definition. $\Psi_{d,\theta}(\mathbb{R}^n)$ consists of pairs (M, ℓ) where $M^d \subset \mathbb{R}^n$ is a smooth manifold and ℓ is a θ-structure on M with a certain topology.

Intuitive description: smooth deformations are ok, and we can also 'push things off to infinity.' A sequence of manifolds converges to a manifold if on all compact subsets K of $\mathbb{R}^n$, they converge to the same germ (i.e. intersection with an open neighborhood of K).

Example. Consider $\{t\} \times \mathbb{R} \subset \mathbb{R}^2$. As $t \to \pm\infty$, $\{t\} \times \mathbb{R} \to \varnothing$. Why is this true? because it is true on a neighborhood of any compact $K \subset \mathbb{R}^2$.

Question. Why the funky definition? What we really want is a topological sheaf $\Psi_d(-)$.

Smooth maps $f : X \to \Psi_d(\mathbb{R}^n)$ where X a smooth manifold. Form the graph

$$\Gamma(f) = \bigcup_{x \in X} \{x\} \times f(x) \subset X \times \mathbb{R}^n$$

f is *smooth* if $\Gamma(f)$ is smooth, closed as a subspace of $\mathbb{R}^n$ and $\pi : \Gamma(f) \to X$ is a submersion.

Example. (n=1, d=0) $f : \mathbb{R} \to \Psi_d(\mathbb{R})$ such that $\Gamma(f)$ looks like one component of the graph of $\tan(x)$. $X = \mathbb{R}$

Non-example. Same thing but with vertical tangency (i.e. near 0, the graph behaves like $\sqrt[3]{x}$)– not a submersion near 0.

2.3 proof of theorem 3.22

(going to do this without θ-structure, and including θ-structures is ‘not that hard’)

Consider $\Psi_{d,\theta}(\mathbb{R}^n)^\circ \subset \Psi_{d,\theta}(\mathbb{R}^n)$ the subspace of submanifolds which pass through the origin. We want a deformation retraction $\Psi_{d,\theta}(\mathbb{R}^n)^\circ \to L^\theta = \mathrm{Gr}_d(\mathbb{R}^n) + \theta$-structure. Dilate/zoom in to origin so that we are left with the tangent plane at the origin.[3]

We have a normal bundle $\nu \to \psi_d(\mathbb{R}^n)^\circ$ where the fiber over M is the normal space to T_0M. We have an embedding

$$\begin{aligned} e : \{\text{open nbd of the zero section}\} &\hookrightarrow \Psi_d(\mathbb{R}^n) \\ (M, v) &\mapsto M + v \end{aligned}$$

So C = complement of the image of e contracts to the empty set $\varnothing$. Therefore we have $\mathrm{Th}(\nu \to \psi_d(\mathbb{R}^n)^\circ) = \psi_d(\mathbb{R}^n)$. We have a diagram in which the lower horizontal arrow is a cofibration (because pushouts preserve cofibrations)

$$\begin{array}{ccc} \nu \cap e^{-1}(C) & \longrightarrow & \nu \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\ C \simeq * & \longrightarrow & \psi_d(\mathbb{R}^n) \end{array}$$

and $\psi_d(\mathbb{R}^n)/C \simeq \psi_d(\mathbb{R}^n) \simeq \nu/\nu \cap e^{-1}(C) \simeq \mathrm{Th}(C)$.

2.4 $\psi_d(n,1) \subset \Psi_d(\mathbb{R}^n)$

Definition. Let $\psi_d(n,1)$ consist of those M which are contained in $\mathbb{R} \times (-1,1)^{n-1}$ and still closed as a subset of $\mathbb{R}^n$.

Give $\psi_d(n,1)$ the subspace topology.

Remark.
- The condition that M be closed as a subset of $\mathbb{R}^n$ means that this is not the same as $\psi_d(\mathbb{R}^n)$ (via the homeomorphism $(-1,1) \cong \mathbb{R}$).
- this looks like cobordisms (all vertical slices are compact)
  $B\mathcal{C}_d \simeq \psi_d(n,1)$
  In order to prove this, we'll use a go between $\mathcal{D}_d(\mathbb{R}^n)$ is a (poset) category with objects subsets of $\mathbb{R} \times \psi_d(n,1) = \{(t, M) \mid t \text{ is a regular value for } x_1 : M \to \mathbb{R}\}$ and morphisms $(t_0, M) \leq (t_1, M)$ where $t_0 \leq t_1$

[3]‘This is the definition of the derivative, you can tell your calculus students that.’

Claim. $B\mathcal{C}_d(\mathbb{R}^n) \simeq B\mathcal{D}_d(\mathbb{R}^n)$

Proof. We will define maps $f : \mathcal{C}_d(\mathbb{R}^n) \to \mathcal{D}_d(\mathbb{R}^n)$, g going the other way.

f on objects: $(t, M) \mapsto M \cap x_1^{-1}(t)$ on morphisms: $(t_0 \leq t_1, M) \mapsto M \cap x_1^{-1}([t_0, t_1])$ shifted left by t_0, then make cylindrical ends (up to homotopy anyways)

g on objects: $M \mapsto M \times \mathbb{R}$ on morphisms: add cylinders to cobordisms on unbounded intervals

Note that $f \circ g = id$

$g \circ f$ takes a morphism in $\mathcal{D}_d$ and 'straightens the ends.' can 'push topology out to infinity, g is a functor on nerves' □

It remains to check that $B\mathcal{D}_d \simeq \psi_d(-,1)$. We have a natural map from LHS to RHS with fibers $B(Reg_M, \leq) \simeq *$ where Reg_M are regular values of x_1 on M.

Formally,

$$\begin{array}{ccc} \partial D^m & \longrightarrow & D^m \\ \big\downarrow \hat{f} & & \big\downarrow f \\ B\mathcal{D}_d(\mathbb{R}^n) & \xrightarrow{u} & \psi_d(n,1) \end{array}$$

For all $a \in \mathbb{R}$, let $U_a = \{x \mid f(x) \text{ has a regular value at } a\}$ these are open sets. Sard's theorem says that any $M \in \psi_d(n,1)$ has a regular value, so $\bigcup U_a = D^m$. Take a finite cover $\bigcup_{i=1}^k U_i = D$. Let λ_i a partition of unity subordinate to the cover $\{U_i\}$ then $g(x) = (\sum_i \lambda_i a_i, f(x))$ where $g : D^m \to B\mathcal{D}_d(\mathbb{R}^n)$ such that $g|_{\partial D^m} \sim \hat{f}$ and $u \circ g = f$.