19 Epilogue

Speaker: Oscar Randall-Williams
[20]

A taste of what else is out there[21]

19.1 Odd dimensions

we don't know the main theorem for odd-dimensional mflds that Oscar would like to know, don't even have a good conjecture

handlebodies Let $V_g = \#_g S^n \times D^{n+1}$ and consider $D^{2n} \subset \partial V_g$. Can consider $B\,\mathrm{Diff}(V_g \,\mathrm{rel}\, D^{2n})$.

Theorem (Perlmutter)**.** *These have homological stability, i.e. $n \geq 4$, $B\,\mathrm{Diff}(V_g \,\mathrm{rel}\, D^{2n}) \to B\,\mathrm{Diff}(V_g \,\mathrm{rel}\, D^{2n})$ is a H_*-iso in a range of degrees $* \leq \frac{g-4}{2}$.*

More general version: $S^p \times S^{q+1}$ with restrictions on p, q.

sort of analogous methods: quadratic modules -¿ wall forms. prove algebraic things are highly connected. Higher form of the whitney trick where intersections occur in 1-manifolds.

Theorem (Botvinnik-P)**.** *(Diffeos that move the boundary classify bundles of mflds with boundary s.t. boundary bundle not trivial)*

There is H_*-iso $\mathrm{hocolim}_g\, B\,\mathrm{Diff}(V_g, D^{2n}) \to Q_0\left(BO(2n+1)\langle n\rangle_+\right)$

surgery: 'push rubbish of the boundary'

correct analogue of what we've done so far? $U_g = \#_g S^n \times S^{n+1}$, $B\,\mathrm{Diff}(U_g, D^{2n+1})$

Theorem (P)**.** *stability holds (same range)*

We *don't* know the stable homology. Can perfectly well form $\Omega^\infty MT(SO(2n+1)\langle n\rangle)$ Ebert showed that there are classes that are nonzero here which are nonzero on all $B\,\mathrm{Diff}$ (or other way around?)

Theorem (Hebestreit-P)**.** *For $n \geq 4, n \neq 7$, have the following monoid under boundary connect sum*

$$\Omega B\left(\bigcup_U B\,\mathrm{Diff}(U, \partial U)\right) \overset{htpyequiv}{\simeq} \Omega B \mathbf{Cob}^{\mathcal{L}}_{2n+1}$$

where U a $(n-1)$-connected, n-parallelizable, $(2n+1)$-dimensional mfld. where RHS still an infinite loop space (but we don't know its homology!...connected to L-theory). RHS is cobordism category where objects/morphisms come with lagrangians...intersection form, still a symmetric monoidal category.

one of the new issues is that manifolds might not intersect, but they might link.

[20]'Most math papers don't get read very much.'
[21]'it turns out there are also odd-dimensional manifolds.'

19.2 Unstable homology

[22] For W^d oriented, closed, d even. $B\operatorname{Diff}(W^d) \subset \mathcal{C}_d(\varnothing, \varnothing) \to \Omega_\varnothing B\mathcal{C}_d^+ \simeq \Omega^\infty MTSO(d)$. Still have this map even when it's not an iso on homology.

$H^*(\Omega^\infty MTSO(d); \mathbb{Q}) = \mathbb{Q}[\kappa_c \mid c$ polynomial in Euler class & pontrjagin classes$] \to H^*(B\operatorname{Diff}^+(W^d); \mathbb{Q})$. the image of this is the *tautological ring*[23] of W $\mathcal{R}^*(W)$. Name comes from algebraic geometry. Stabilization results don't help with ring-theoretic questions (e.g. ideals, nilpotence). hom stab tells us about additive structure (one dim at a time), not really about multiplicative structure.

Theorem (Grigoriev (n odd), RW (n even))**.** *$\mathcal{R}^*(W_g^{2n})$ is a finitely generated $\mathbb{Q}$-algebra.*

Now you can ask ring-theoretic questions.

Theorem (Galatius-Grigoriev-R-W)**.** *For n odd.*

$g = 0$ $R^*(W_0^{2n} = S^{2n}) = \mathbb{Q}[\kappa_{ep_1}, \ldots, \kappa_{ep_k}]$ *has Krull dimension n*
$g = 1$ $R^*(W_1^{2n} = S^n \times S^n)/\sqrt{0} = \mathbb{Q}$ *has Krull dimension 0*
$g > 1$ $R^*(W_g^{2n})/\sqrt{0} = \mathbb{Q}[\kappa_{ep_1}, \ldots, \kappa_{ep_k}]$ *has Krull dimension $n-1$*

Tool: theorem about equivariant cohomology and fixed point set (??)

Fact. The cohomology of a lie group injects into the cohomology of its maximal torus.

19.2.1 Surfaces

$\Sigma_{g(,1)}$ surfaces and $\Gamma_{g(,1)} = \pi_0 \operatorname{Diff}^+(\Sigma_{g,1})$ mapping class groups

homological stability: can phrasing of the *relative homology* vanishes $H_d(\Gamma_{g,1}, \Gamma_{g-1,1}) = 0$ $d \leq \frac{2g-2}{3}$ (Harer, Ivanov, Boldsen).

Slogan: relative homology measures failure of hom stab. Failure of hom stab holding satisfies homological stability in a different way

Theorem (Galatius-Kupers-RW)**.** *There are maps*

$$\phi_* : H_{d-2}(\Gamma_{g-3,1}, \Gamma_{g-4,1}) \to H_d(\Gamma_{g,1}, \Gamma_{g-1,1})$$

which is epi for $d \leq \frac{3g-1}{4}$, iso for $d \leq \frac{3g-5}{4}$. rationally, epi for $d \leq \frac{4g-1}{5}$, iso for $d \leq \frac{4g-6}{5}$

in the category of E_2 algebras, what's the minimal way in which you can build this out of cells.

$$\coprod_{g \geq 0} B\Gamma_{g,1}$$

attaching finitely many cells of each dimension doesn't break hom stability. (something about Fred Cohen's thesis)

Theorem (Chan-Galatius-Payne)**.** $\dim H^{4g-6}(\Gamma_g; \mathbb{Q}) > (1.3247)^g + $ *const.*

[22]'unstable things exist as well'
[23]'people call it that..people? me'

How to prove: methods from algebraic geometry.

$B\Gamma_g \simeq \mathcal{M}_g$ Riemann's moduli space has a compactification (Deligne-Mumford) $\subset \overline{M}_g$. cohomology comes from intersections near the boundary. [24]

what we don't know homological stability for 4-manifolds. homeomorphisms in high dimensions: same story (due to Sander Kupers). htpy type of the topological cobordism category (what are topological microbundles).

$\pi_1 = 0$ is only used in hom stab (whitney trick)

non-simply-connected: (count intersections in group ring of π_1, algebraically harder, look at work of Nina F.–Oscar's student)

References

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[Kup17] Alexander Kupers. Some finiteness results for groups of automorphisms of manifolds. *Geom. Topol.*, 2017.

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[24]'maybe I shouldn't try to describe the proof, given that Søren's right there'