18 Rational homotopy of $B\,\mathrm{aut}_\partial(M_{g,1})$

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Following [BM14]

Worlds: topology/manifolds ⇝ algebra (dglas) ⇝ combinatorics, graph complexes

18.1 Setup

Topology: Here $M_{g,1} = W_{g,1} = \#_g S^d \times S^d \setminus D^{2d}$

In particular dimension $2d$, $d-1$-connected, and $\partial M_{g,1} \cong S^{2d-1}$. Assume $d \geq 3$ (many things go through for $d \geq 2$ but not all.)

We want to study $X_g = B\,\mathrm{aut}_\partial(M_{g,1})$ = homotopy autoequivalences fixing ∂ pointwise. And denote $\tilde{X}_g = B\,\mathrm{aut}_{\partial,\circ}(M_{g,1})$. Recall from the last talk that we have a fiber sequence (from SES of groups) $\tilde{X}_g \to X_g \to B\pi_1 X_g$

Algebra:

$$\begin{aligned} V_g = s^{-1}\tilde{H}_*(M_{g,1};\mathbb{Q}) \in \mathbf{Sp} = \mathbf{Sp}_{d-1}^{2(d-1)} &= \text{ antisymmetric qr v.s. with inner product of degree } 2(d-2) \\ = s^{d-1}H_g \qquad &= (H^{\oplus g}, M, q) \text{ geometric quadratic module} \end{aligned}$$

$\Gamma_g = \mathrm{Aut}(H_g, M, q)$

$\mathcal{G}_g = \mathrm{Der}^+_{w_g}(\mathbb{L}V_g)$

$\tilde{X}_g, (M_{g,1}, \partial M_{g,1} \cong S^{2d-1})$ has a rational model given by $\mathcal{G}_g, (\mathbb{L}V_g, w_g)$

We have a *stabilization maps*, which can mean anything from

$$\begin{aligned} \mathrm{aut}_\partial(M_{g,1}) &\to \mathrm{aut}_\partial(M_{g+1,1}) \\ X_g &\to X_{g+1} \\ H_g &\to H_{g+1} \\ \Gamma_g &\to \Gamma_{g+1} \end{aligned}$$

18.2 Main theorems

Theorem. *(7.6)*
$\sigma_k : H_k(X_g;\mathbb{Q}) \to H_k(X_{g+1};\mathbb{Q})$ is an isomorphism for $g > 2k+4$

Theorem. *$H^*(X_\infty;\mathbb{Q}) \cong H^*(\Gamma_\infty;\mathbb{Q}) \otimes H^*_{CE}(\mathcal{G}_\infty)^{\Gamma_\infty}$ where $X = \mathrm{hocolim}_g X_g$, $\Gamma_\infty = \mathrm{colim}_g \Gamma_g$.*

Theorem. *$C^{CE}_*(\mathcal{G}_\infty)_{\Gamma_\infty} \cong (\Lambda \mathcal{G}^d(0), \partial)$ where Λ means 'free graded-commutative' and $\mathcal{G}$ is a graph complex*

Recall: fiber sequence $\tilde{X}_g \to X_g \to B\pi_1(X_g)$ where (1) RHS is geometric and (2) is algebraic

BM 13, 2.12 There's an exact sequence (recall Shruthi's talk) $0 \to K \to \pi_1 X_g \to \Gamma_g \to 0$ where LHS is finite and RHS is arithmetic.

The upshot is that $\pi_1 X_g$ is 'close' to Γ_g.

$\implies$ (7.8) is rationally perfect $\implies$ no extension problem on modules.

Stasheff, 3.10, 3.11 $\pi_*^{\mathbb{Q}}(\tilde{X}_g) \cong \mathcal{G}_g$ as graded Lie algebras.
Definition (bad): $\pi_*^{\mathbb{Q}}(X) = \pi_{*+1}X \otimes \mathbb{Q}$. Moreover, $\pi_1(X_g)$-equivariant. Action on LHS: π_1 action on π_k. Action on RHS: by action of Γ_g.

We glue these together using the universal cover spectral sequence.

18.3 Σ-mod and $\mathcal{L}ie$ operad

Definition. $\mathcal{C}$ is a Σ-module if for each $k \geq 0$, have $\mathcal{C}(k)$ a graded vector space with a Σ_k-action.

An *operad* is a "composable" Σ-module. [picture of composition in operads where elements are represented as trees]

A *cyclic operad* is an operad with an action of Σ_{k+1} on $\mathcal{C}(k)$.

[picture of an element of $\mathcal{C}(k)$ as having k inputs $+1$ output, and Σ_{k+1} acting on the 'leaves']

The $\mathcal{L}ie$ operad $\mathcal{L}ie(k)$ is spanned by Lie polynomials in $x_1, \ldots, x_k$.

Example. $\mathcal{L}ie(1) = \mathbb{Z}\{x_1\}$
$\mathcal{L}ie(2) = \mathbb{Z}\{[x_1, x_2]\}$
$\vdots$

Proposition. *$\mathcal{L}ie$ is a cyclic operad. Write $\mathcal{L}ie((n)) = \mathcal{L}ie(n-1)$*

We can associate a Schur functor to a Σ-module $\mathcal{C}$:

$$C : \mathbf{gr.\ Ab} \to \mathbf{gr.\ Ab}$$
$$H \mapsto \oplus_{k=0}^{\infty} \mathcal{C}(k) \otimes_{\Sigma_k} H^{\otimes k}$$

Examples.

- sH. Then $S(k) = \begin{cases} \mathbb{Z}[-1] & k = 1 \\ 0 & k \neq 1 \end{cases}$
- $\bigwedge H \implies \Lambda(k) = sgn_{\Sigma_k}$
  $C_*^{CE}(H) = \bigwedge sH$

Definition. $\mathcal{L}ie((V)) = s^{-2(d-1)} \oplus_{n \geq 2} \left(\mathcal{L}ie((n)) \bigotimes_{\Sigma_n} V^{\otimes n}\right)$

Proposition (6.6)**.** *Have an iso of functors $Sp \to$ **gr v s** $\mathrm{Der}_w(\mathbb{L}V) \cong \mathcal{L}ie((V))$*

Idea $\mathrm{Hom}(V, \mathbb{L}V) \cong V^* \otimes \mathbb{L}V \cong V \otimes \mathbb{L}V$.
Recall: $V = V_g$, $\mathcal{G}_g \cong \mathcal{L}ie^+((V_g))$. The upshot is that $\mathcal{G}_g$ trivial below degree $d-1$.

18.4 Stability

Recall our tool is universal cover spectral sequence applied to $\tilde{X}_g \to X_g \to B\pi_1 X_g$

Universal cover spectral sequence (is this right?) $E^2_{p,q} = H_p(\pi_1(X_g); H_q(\tilde{X}_g;\mathbb{Q})) \implies H_{p+q}(X_g;\mathbb{Q})$

Strategy: show stability/isomorphism on E^2 page.

Step 1: fiber $H_q(\tilde{X}_g;\mathbb{Q}) \overset{2.3}{\cong} H_q^{CE}(\lambda(\tilde{X}_g))$ because Quillen SS collapses, and RHS is formal, so $\cong H_q^{CE}(\pi_*^{\mathbb{Q}}(\tilde{X}_g)) \overset{5.5}{\cong} H_q^{CE}(\mathcal{G}_g)$

so $E^2_{p,q} \cong H_p(\pi_1(X_g); H_q^{CE}(\mathcal{G}_g))$ π_1-equivariant, compatible with σ.

Step 2: Recall SES of groups $0 \to k \to \pi_1 X_g \to \Gamma_g \to 0$ where the action by (-) is given by projection. Therefore (using finiteness of kernel) $E^2_{p,q} \cong H_p(\Gamma_g; H_q^{CE}(\mathcal{G}_g))$

Step 3: (Stability) $\sigma : H_p(\Gamma_g; H_q^{CE}(\mathcal{G}_g)) \to H_p(\Gamma_{g+1}; H_q^{CE}(\mathcal{G}_g))$ isomorphism for $g > 2p+2q+4$.

18.5 Polynomial functors

Definition. A functor $P : \mathbf{Ab} \to \mathbf{Ab}$ is polynomial of degree $\leq \ell$ where $P(H) = \bigoplus_{k=0}^{\ell} P(k) \otimes_{\Sigma_k} H^{\otimes k}$ Schur functor that vanishes above level ℓ

Why do we care?

Theorem (Charney)**.** *H_g, Γ_g as before, P polynomial of degree $\leq \ell$.*
$\sigma : H_p(\Gamma_g; P(H_g)) \to H_p(\Gamma_{g+1}; P(H_{g+1})$ is iso for $g > 2p + \ell + 4$.

Theorem. *Vanishing theorem (7.5)*

$$H^k(\Gamma_g;\mathbb{Q}) \otimes P(H_g \otimes \Gamma_g) \cong H^k(\Gamma_g; P(H_g^{\mathbb{Q}} \otimes \mathbb{Q}))$$

isomorphism for $g > 2k + \ell + 4$

Lemma. *(7.2)*
Fix q

$$C : H_g \mapsto C_q^{CE}(\mathcal{G}_g) = C_q^{CE}(\operatorname{Der} \mathbb{L} S^{d-1} H_g) = C_*^{CE}(\mathcal{L}ie((S^{d-1}H_g)))$$

is polynomial of degree $\leq \lfloor \frac{3q}{d} \rfloor$

Idea: C is "Taylor."
So what? $H^*(\Gamma_g;\mathbb{Q}) \otimes (H^*_{CE}(\mathcal{G}_g))^{\Gamma_g} \cong H^*(\Gamma_g; C^q_{CE}(\mathcal{G}_g))$ in a range.
prop 7.11: works for H_q^{CE} replacing C_q^{CE} for $g > 2p + 2q + 4$.

18.6 Stable cohomology

18.2 For each q E^2 of RHS $\implies$ LHS.

Fact. 0. (8.5) $H^*(X_\infty;\mathbb{Q})$ free graded commutative
1. (Borel) $H^*(\Gamma_\infty;\mathbb{Q}) \cong \mathbb{Q}[x_1, x_2, \ldots]$
$$|x_i| = \begin{cases} ?? & d \text{ odd} \\ ?? & d \text{ even} \end{cases}$$
2. (GRW) $H^*(B\operatorname{Diff}_\partial(M_{g,1}))$
3. by hand, $H^*(\Gamma_\infty;\mathbb{Q}) \to H^*(B\operatorname{Diff}_\partial(M_{g,1}))$ is injective on indecomposables, hence so is $H^*(\Gamma_\infty;\mathbb{Q}) \to H^*(X_\infty)$.
$\therefore$ no differentials

18.7 Graph complex

$\mathcal{G}^d(0)$ graph complex. $d = 1$ (otherwise just contributes to a degree shift)

$\mathcal{G}^d(0)_k = $ cnt (??) graphs of k vertices, each of degree $k \geq 3$, with orientation on vertices and edges (??) and decoration of vertices by elements of $\mathcal{L}ie$ operad.

The differential $\mathcal{G}^d(0)_k \to \mathcal{G}^d(0)_{k-1}$ is given by contraction of edges and summing over all of these with signs.

$$\begin{aligned}
C_*^{CE}(\mathcal{G}_\infty)_{\Gamma_\infty} &\cong \operatorname*{colim}_g C_*^{CE}(\mathcal{G}_g)_{\Gamma_g} \\
&= \operatorname*{colim}_g C_*^{CE}(\mathcal{L}ie((V_g)))_{\Gamma_g} \\
&\cong \operatorname*{colim}_g \left(\bigwedge s\mathcal{L}ie((V_g))\right)_{\Gamma_g} \\
&\cong \operatorname*{colim}_g \left(\bigwedge s\widetilde{\mathcal{L}ie}\right)(n) \otimes_{\Sigma_n} (V_g^{\otimes n})_{\operatorname{Aut} V_g} \\
&\cong \bigwedge \mathcal{G}^d(0)
\end{aligned}$$

commuting colimits and coinvariants

some 'fundamental theorem'

$\operatorname{colim}_g (V_g^{\otimes n})_{\operatorname{Aut} V_g} \cong M_n \otimes sgn_{\Sigma_n}$ at level k where M_n is matching (??) on n letters