# 17 Rational homotopy theory of automorphism monoids of manifolds

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## 17.1 Introduction

Notation $\mathrm{aut}(X) = h\,\mathrm{Aut}(X)$ = topological monoid of self homotopy equivalences of $X$. $\mathrm{aut}_{\partial,\circ,*}$ where the subscripts mean [fixing] boundary, component of identity, pointed respectively.

$\mathrm{Fib}(B;X)$ = fibrations over $B$ with fiber $X$

$\mathbf{Top}^{\geq 2}, \mathbf{Mon}^{\geq 2}$ topological spaces, manifolds which are 1-connected.

Before: $B\,\mathrm{Diff}^{\partial}(W_{g,1})$ Today: $B\,\mathrm{aut}^{\partial}(W_{g,1})$

Recall Let $M \in \mathbf{Man}$ compact, there is a fiber bundle with fiber $M$: $E_M \to B_M$ which is universal, i.e.

$$[B, B_M] \simeq \mathrm{Fib}(B;M), \qquad \text{(smooth fiber bundles)}$$

$\varphi \mapsto \varphi^*(E_M)$ is a bijection for any manifold $B$. Furthermore previous is weakly equivalent $B\,\mathrm{Diff}_*(M) \to B\,\mathrm{Diff}(M)$

**Theorem.** *$X \in \mathbf{Top}$ a finite CW-complex. Then there is a universal $X$-fibration $E_X \to B_X$ which is universal*

$$[B, B_X] \simeq \mathrm{Fib}(B;X)$$

*which is a bijection for any $B$ CW complex. Furthermore * is weakly equivalent $B\,\mathrm{aut}_*(X) \to B\,\mathrm{aut}(X)$*

**Theorem.** *$L$ a cofibrant dgla. There is a universal $L$-fibration which is universal $E_L \to B_L$ such that*

$$[B, B_L] \simeq \mathrm{Fib}_{\mathbf{dgla}}(B;L)$$

*a bijection for $B$ any cofibrant dgla. ...* $\mathrm{Der}^+(L) \to (\mathrm{Der}\,L \,/\!\!/\, ad\ L)^+$

**Theorem.** *$M^n \in \mathbf{Man}^{\geq 2}$ compact, $\partial M = S^{n-1}$ then dgla model for $B\,\mathrm{aut}_{\partial,\circ}(M)$ is*

$$(\mathrm{Del}_w^{\oplus}\,\mathbb{L}(V := s^{-1}\tilde{H}(M;\mathbb{Q}); [\delta, -]))$$

*where $w$ canonical element of $V^{\otimes 2}$. $s^{-1}$ desuspension, $\oplus$ with dotted circle*[16]

*Fact.* $M = W_{g,1} \implies [\delta, -] = 0$

## 17.2 $\mathbb{Q}$-homotopy theory of Quillen

**Definition.** $X \in \mathbf{Top}$ is *rational* if $\pi_* X$ are $\mathbb{Q}$-vector spaces.

$\forall X \in \mathbf{Top}^{\geq 2}$ $\exists X \to X_{\mathbb{Q}}$ such that $X_{\mathbb{Q}}$ rational and $\pi_* i : \pi_* X \otimes \mathbb{Q} \to \pi_* X_{\mathbb{Q}}$ iso, $i$ universal. (rationalizaton is functorial)

**Definition.** $\varphi : X \to Y \in \mathbf{Top}^{\geq 2}$ a *rational equivalence* if TFAE holds

- $\pi_* \varphi \otimes \mathbb{Q}$ iso

[16] bourbaki's way of saying not the same as before.

- $H_*(\varphi;\mathbb{Q})$ iso
- $H^*(\varphi;\mathbb{Q})$ iso
- $\varphi_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is a w.h.e.

**Theorem** (Quillen). $\exists\lambda : \mathbf{Top}^{\geq 2} \to \mathbf{dgla}^{\geq 1}$ *such that* $\mathbf{Top}^{\geq 2}([\simeq_{\mathbb{Z}}]^{-1}) \xrightarrow{\simeq} \mathbf{dgla}^{\geq 1}([\simeq]^{--1})$ *Furthermore,* $\pi_*(X) \otimes \mathbb{Q}[-1] \simeq H_*(\lambda(X))$ *as graded Lie algebras.*

$H_*(X;\mathbb{Q}) \cong H_*^{CE}(\lambda(X))$ *as graded coalgebras.*

*Remark.* $\lambda$ is non-tractable.

$N_*Prim(\mathbb{Q}[GX]_I^\wedge)$

## 17.3 Fibrations of DGLAs

**Definition.** $L$ get $L_\bullet^+ = \begin{cases} L_i & i \geq 2 \\ \ker L_1 \to L_0 & i = 1 \\ 0 & i \leq 0 \end{cases}$

For $f : L \to L' \in \mathbf{dgla}$ form chain complex

$$\mathrm{Der}_f(L, L')_p = \{\theta : L_\bullet \to L'_{\bullet+p} \mid \theta[x,y] = [\theta x, y] + (-1)^{f\theta}[fx, \theta y]\}$$

differential $D\theta = d'_L \circ \theta - (-1)^\theta \theta \circ d_L$

$Der(L) = Der_{id}(L, L)$ is a dgla. $[\theta, \tau] = \theta \circ \tau - (-1)^{\theta\tau}\tau \circ \theta$

$ad : L \to \mathrm{Der}\, L$, $x \mapsto [x, -]$ (derivation follows from jacobi identity)

$\mathrm{Der}\, L \,/\!/\, adL$= mapping cone (ad) = $sL \oplus \mathrm{Der}\, L$ is a dgla

**Definition.** A *L-fibration* is a surjective $\pi : E \to B$ map of dglas with $L \xrightarrow{\simeq} \ker\pi$. An equivalence of $L$-fibrations is an equivalence over $B$.[17]

**Corollary.** *$X$ finite CW complex,* $\mathbb{L}_X$ *cofibrant model for* $\lambda(X)$*. Then* $(\mathrm{Der}\,\mathbb{L}_X)^+ \to (\mathrm{Der}\,\mathbb{L}_X \,/\!/\, adX)^+$ *is a dgla model* $B\,\mathrm{aut}_*(X)\langle 1\rangle \to B\,\mathrm{aut}(X)\langle 1\rangle$.

This is a corollary of theorem from last section, a theorem classifying $L$-fibrations

Heuristic Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is $\mathrm{Der}(\mathfrak{g})$

## 17.4 Explicit model for $B\,\mathrm{aut}_{\partial,\circ}(W_{g,1})$

Interesting, we won't use it again

**Theorem** (Lupton-Smith, 2007). *$f : X \to Y$ map of simply-connected spaces ($\in \mathbf{Top}^{\geq 2}$) with Lie model* $\mathbb{L}_X \to \mathbb{L}_Y$*. Then* $\pi_k(\mathrm{Map}_*(X,Y); f) \xrightarrow{\beta} H_k(\mathrm{Der}_\varphi(\mathbb{L}_X, \mathbb{L}_X))$ *is a bijection for $k = 1$ and for $k \geq 2$ an isomorphism of $\mathbb{Q}$-vector spaces.*

*When $X = Y, f = id_X$, then $\beta$ is $\cong$ of graded Lie algebras.*

Problems: models for $\mathbb{L}_X$ can be complicated (at least in Quillen's work)

Fix $M^n$ compact, $\partial M = S^{n-1}$, $M \in \mathbf{Man}^{\geq 2}$. Then get a fiber sequence $\mathrm{aut}_{\partial,\circ}(M) \to \mathrm{aut}_\partial(M) \to \pi_0\,\mathrm{aut}_\partial(M)$ which we can deloop $B\,\mathrm{aut}_{\partial,\circ}(M) \to B\,\mathrm{aut}_\partial(M) \to B\pi_0\,\mathrm{aut}_\partial(M)$. Want to understand the middle. understand RHS on a case-by-case basis

[17] 'forgive me for calling a lie algebra $E$ for once in my life'

GOAL for today is to understand LHS.

Let $\{\alpha_i\}$ homogenous basis for $V = s^{-1}\tilde{H}(M;\mathbb{Q})$. $\{\alpha_i^{\#}\}$ dual basis w.r.t. *sign*$\cdot$( intersection form extended by 0)$\circ$ $(s \otimes s)$. $w = \sum_i \alpha_i \otimes \alpha_i^{\#} = \frac{1}{2}\sum_i [\alpha_i, \alpha_i^{\#}] \in \mathbb{L}V$ an element of $V^{\otimes 2}$ where $[-,-]$ is the commutator with respect to $\otimes$.

dgla model for $S^{n-1} = \partial M \hookrightarrow M$

**Theorem** (Stasheff 1983)**.** *In the above setting, $\exists \delta$ differential on $\mathbb{L}V$ such that*

*(1) $(\mathbb{L}V, \delta)$ models $M$*
*(2) $W \in \mathbb{L}V$ is a cycle representing $(-1)^n([\partial M \hookrightarrow M])$*

$$\text{Write } \operatorname{Der}_w^{\oplus}(\mathbb{L}V)_p := \begin{cases} \{\theta \in \operatorname{Der}(\mathbb{L}V)_p \mid \theta(w) = 0, \text{ if } p = 1, [\delta, \theta] = 0\} & p \geq 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{This is a dgla.}$$

**Theorem.** *In the above setting $B\operatorname{aut}_{\partial,\circ}(M)$ is dgla modeled by* $(\operatorname{Der}_w^{+}(\mathbb{L}V), [\delta, -])$

**Corollary.** *When $M = W_{g,1}$, $[\delta, -] = 0$*

$$\therefore \pi_*\left(B\operatorname{aut}_{\partial,\circ}(W_{g,1})\right) \otimes \mathbb{Q}[-1] \cong \operatorname{Der}_w^{+} \mathbb{L}V$$

*as graded Lie algebras.*[18]

*Proof.* (of theorem)

Let $\rho$ generator of degree $n-2$. Let $\mathbb{L}(\rho)$ free Lie algebra generated by $\rho$

$$\mathbb{L}(\rho) \xrightarrow{\rho \mapsto (-1)^n w} (\mathbb{L}V, \delta)$$

[19]

factor

$$\begin{array}{ccc} \mathbb{L}_{\partial M} = \mathbb{L}(\rho) & \longrightarrow & (\mathbb{L}V, \delta) = \mathbb{L}_M \\ & \searrow & \uparrow {\scriptstyle \gamma \mapsto 0, \simeq} \\ & & (\mathbb{L}(V, \rho, \gamma); \delta) = \tilde{\mathbb{L}}_M \end{array}$$

where bottom has same dgla structure. By theorem 3.4 (how to model $A \hookrightarrow X \in \mathbf{Top}^{\geq 2}$) $\operatorname{Der}^{+}(\tilde{L}_M; \mathbb{L}_{\partial M})$ (derivations that vanish on latter) models $B\operatorname{aut}_{\partial,\circ}(M)$ and former is (quasi)-isomorphic to $(\operatorname{Der}_w^{+}(\mathbb{L}V), [\delta, -])$. ∎

[18] what are formal spaces?
[19] (another result: inclusion of two simply-connected spaces is given by (?) under some cofibrancy conditions)