Talbot 2019 Talk 16

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1 THE CONNECTION...

...between last talk and everything before it.

Once again we consider a spin manifold W^d with $\partial W^d = M^{d-1}$. Why did we want W a spin manifold?—Because it allowed us to define the "index difference" operator

$$\mathcal{R}^+(W)_h \xrightarrow{\text{inddiff}} \Omega^{\infty+d+1}\mathsf{KO}$$

as seen in the previous talk.

1.1 THE ACTION OF THE DIFFEOMORPHISM GROUP

We have an action $\mathrm{Diff}_\partial(W) \to \mathrm{Aut}(\mathcal{R}^+(W)_h)$ by pulling back metrics along diffeomorphisms. For the tangential structure $\theta = \mathrm{Spin}$ this gives us a diagram

$$\begin{tikzcd}
B\mathrm{Diff}_\partial(W) \arrow[r] \arrow[d, "\psi"] & B\,\mathrm{Aut}(\mathcal{R}^+(W)_h) \\
\Omega^\infty \mathsf{MT}\theta \arrow[ur, dashed] &
\end{tikzcd}$$

where ψ is the map given as the composition of $B\mathrm{Diff}_\partial(W) \to \Omega B\mathcal{C}_\theta \simeq \Omega^\infty \mathsf{MT}\theta$, which was discussed in the first part of this workshop (the map from $B\mathrm{Diff}_\partial(W)$ in the first talk, the latter one is [GMTW09, Main Theorem]). We want the indicated dashed map to achieve our goal of understanding $\mathcal{R}^+(W)_h$ better by factorising a known map through it.

Let's study π_1 of the horizontal arrow:

Theorem (Abelianness theorem, [BER17, Thm. 4.1.2])**.** *Let W be compact, spin, simply connected with $\partial W = S^{d-1}$, $d \geq 5$. Assume that W is* Spin*-bordant to $D^d \operatorname{rel} \partial$ and let $h_\circ = h_\circ^{d-1}$ be the round metric on S^{d-1}. Then the image of*

$$\pi_0(\mathrm{Diff}_\partial(W)) \longrightarrow \pi_0(\mathrm{Aut}(\mathcal{R}^+(W)_{h_\circ}))$$

is abelian.

We will prove this theorem using the following rather abstract statement about topological categories, which is of Eckmann–Hilton flavour:

Lemma ([BER17, Lem. 4.1.3])**.** *Let $\mathcal{C}$ be a (non-unital) topological category with* $\operatorname{Obj}\mathcal{C} = \mathbb{Z}$*, and suppose we have a topological group G acting equivariantly on the morphism spaces $\mathcal{C}(m,n)$ for all $m, n \in \mathbb{Z}$ such that*

i) $\mathcal{C}(m,n) = \emptyset$ *when* $n \leq m$

ii) *For all $m \neq 0$, there is $u_m \in \mathcal{C}(m, m+1)$ such that the composition maps*

$$\begin{aligned} u_m \circ -\colon \mathcal{C}(m+1,n) &\longrightarrow \mathcal{C}(m,n) \qquad \text{for } n > m+1 \\ - \circ u_m\colon \mathcal{C}(n,m) &\longrightarrow \mathcal{C}(n,m+1) \qquad \text{for } n < m \end{aligned}$$

are homotopy equivalences.

iii) *There exists $x_0 \in \mathcal{C}(0,1)$ such that the composition maps*

$$\begin{aligned} x_0 \circ -\colon \mathcal{C}(1,n) &\longrightarrow \mathcal{C}(0,n) \qquad \text{for } n > 1 \\ - \circ x_0\colon \mathcal{C}(n,0) &\longrightarrow \mathcal{C}(n,1) \qquad \text{for } n < 0 \end{aligned}$$

are homotopy equivalences.

iv) *G acts trivially on $\mathcal{C}(m,n)$ unless $m \leq 0$ and $n \geq 1$.*

Then the action of G commutes up to homotopy on $\mathcal{C}(0,1)$.

Proof. Let $f, g \in G$ be elements of the group G. Then there are $y_f \in \mathcal{C}(-1,0)$ and $z_f \in \mathcal{C}(1,2)$ such that

$$y_f \circ x_0 \sim u_{-1} \circ f(x_0) \in \mathcal{C}(-1,1) \qquad \text{and} \qquad x_0 \circ z_f \sim f(x_0) \circ u_1 \in \mathcal{C}(0,2)$$

where "$\sim$" means being equivalent on π_0. This can be achieved by letting y_f correspond to the path component of the image of x_0 under the homotopy equivalences

$$x_0 \in \mathcal{C}(0,1) \xrightarrow{u_{-1}\circ -} \mathcal{C}(-1,1) \xrightarrow{f} \mathcal{C}(-1,1) \xleftarrow{-\circ x_0} \mathcal{C}(-1,0)$$

For z_f one uses an analogous argument.

Now we show that for $f, g \in G$ the maps

$$fg(- \circ u_{-1} \circ x_0 \circ u_1),\ gf(- \circ u_{-1} \circ x_0 \circ u_1)\colon \mathcal{C}(-2,1) \longrightarrow \mathcal{C}(-2,2)$$

are homotopic.

$$\begin{aligned} fg(- \circ u_{-1} \circ x_0 \circ u_1) &= f(- \circ u_{-1} \circ gx_0 \circ u_1) \\ &\simeq f(- \circ u_{-1} \circ x_0 \circ z_g) \\ &= (- \circ u_{-1} \circ fx_0 \circ z_g) \\ &\simeq (- \circ y_f \circ x_0 \circ z_g) \\ &\simeq \cdots \\ &= gf(- \circ u_{-1} \circ x_0 \circ u_1) \end{aligned}$$

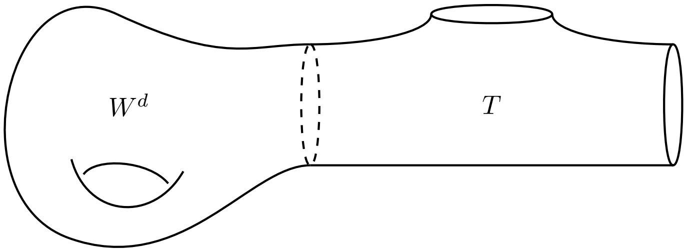

Figure 1: The manifold W^d with T glued onto it

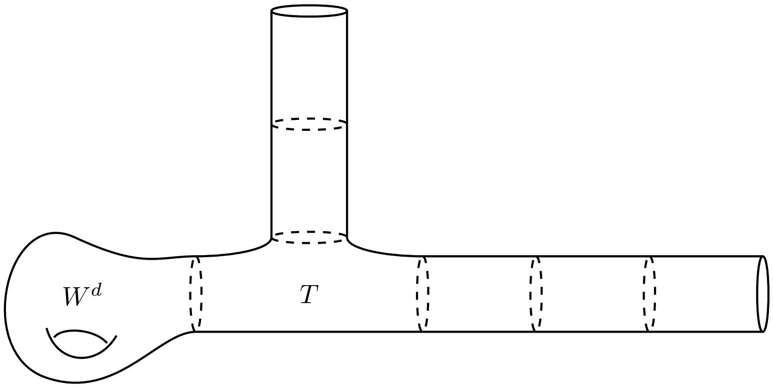

Figure 2: the manifold, that has $\mathcal{C}(-2,3)$ as its space of psc metrics

Finally, $(- \circ u_{-1} \circ x_0 \circ u_1)\colon \mathcal{C}(-2,-1) \to \mathcal{C}(-2,2)$ is a homotopy equivalence and therefore the two maps $f, g\colon \mathcal{C}(-2,2) \to \mathcal{C}(-2,2)$ are homotopic. Using the the diagram

$$\begin{tikzcd}
\mathcal{C}(0,1) \arrow[r, "u_{-2}\circ u_{-1}\circ - \circ u_1", "\simeq"'] \arrow[d, "h"] & \mathcal{C}(-2,2) \arrow[d, "h"] \\
\mathcal{C}(0,1) \arrow[r, "u_{-2}\circ u_{-1}\circ - \circ u_1", "\simeq"'] & \mathcal{C}(-2,2)
\end{tikzcd}$$

we see that $fg \simeq gf\colon \mathcal{C}(0,1) \to \mathcal{C}(0,1)$ as required. ☐

Proof of the Abelianness Theorem. We define need to define $\mathcal{C}(m,n)$. As the group we choose $G = \mathrm{Diff}_d(W)$ and we obviously *want* $\mathcal{C}(0,1) \simeq \mathcal{R}^+(W)_{h_\circ}$.

Now let $T = [0,1] \times S^{d-1} \setminus \operatorname{int} D^d$ be a cylinder, with a disc removed. Gluing T and W we get $V = W^d \cup_{S^{d-1}} T$ as in fig. 1. Via the Corbordism Theorem from the previous talk we have $\mathcal{R}^+(W)_{h_\circ} \simeq \mathcal{R}^+(V)_{h_\circ,h_\circ}$ (by gluing the torpedo metric back into the removed disk), hence we set $\mathcal{C}(0,1) = \mathcal{R}^+(V)_{h_\circ,h_\circ}$ and let G act on it by extending the action trivially onto T.

To define $\mathcal{C}(m,n)$ in general we attach cylinders of unit length to the two boundary components as follows: For $m \leq 0$ in the first argument we attach the according number of cylinders to the "top" of V, for $n \geq 1$ in the second argument to the right, see fig. 2 for the idea. If $m \geq n$, we consider the empty set and if both are negative or positive, we only consider the respective cylinder. In all those cases $\mathcal{C}(m,n)$ is the space of psc metrics on that manifold (and in particular empty in some cases), see [BER17, p. 42] for the actual formula.

Composition is simply given by gluing. The group $G = \mathrm{Diff}_\partial(W)$ acts on $\mathcal{C}(n,m)$ trivially unless $n \leq 0, m \geq 1$ – in that case the action comes from extending by the identity from W.

We let $u_m \in \mathcal{C}(m, m+1) = \mathcal{R}^+(S^{d-1} \times [m, m+1])$ be the cylinder metric $u_m = dt^2 + h_\circ^{d-1}$, which fulfils assumption (ii) of the Lemma.

Using the Cobordism Theorem[1] one finds $x_0 \in \mathcal{C}(0,1)$ fulfilling assumption (iii), since V is cobordant relative to its boundary to $S^{d-1} \times [0,1]$. □

1.2 Maps into the space of psc metrics $\mathcal{R}^+(W)_h$

Let's get back to our diagram from the beginning:

$$\begin{tikzcd}
B\mathrm{Diff}_\partial(W) \arrow[r] \arrow[d, "\psi"] & B\,\mathrm{Aut}(\mathcal{R}^+(W)_h) \\
\Omega^\infty \mathsf{MT}\,\mathrm{Spin} \arrow[ur, dashed, "\exists?"] &
\end{tikzcd}$$

We just showed that the horizontal map has abelian image in π_1; in particular this implies, that the commutator subgroup of $\pi_0(\mathrm{Diff}_\partial(W))$ lies in the kernel of the horizontal map. This implies, that a suitable dashed map exists if ψ is acyclic (see [HH79, Prop. 3.1]). This idea comes up in the proof of the following central theorem:

Theorem ([BER17, Thm. B])**.** *Let W^d be spin and $d = 2n \geq 6$. Fix $h \in \mathcal{R}^+(M^{d-1})$ and $g_0 \in \mathcal{R}^+(W^d)_h$. Then there is a map ρ such that the composition*

$$\Omega^{\infty+1}\mathsf{MT}\,\mathrm{Spin}(2n) \xrightarrow{\quad\rho\quad} \mathcal{R}^+(W)_h \xrightarrow{\mathrm{inddiff}_{g_0}} \Omega^{\infty+2n+1}\mathsf{KO}$$

is weakly homotopic[2] to an infinite loop space version of the $\hat{A}$-genus

$$\Omega^{\infty+1}\mathsf{MT}\,\mathrm{Spin}(2n) \xrightarrow{\Omega\hat{A}_{2n}} \Omega^{\infty+2n+1}\mathsf{KO}$$

Remark. There is also a version of Theorem B for odd dimensions, which is called Theorem C in [BER17] and can be derived from Theorem B via the techniques mentioned in the previous talk.

Remark. Recall that Theorem A says that $\mathrm{inddiff}_{g_0}$ is surjective in $\pi_k \otimes \mathbb{Q}$. It is derived from Theorem B via a computation of $\Omega\hat{A}_{2n}$ on $\pi_k \otimes \mathbb{Q}$. For this workshop we took a different approach of proving Theorem A, which allows us to circumvent proving Theorem B in its full generality.

However, the following question has to be answered: What is the $\hat{A}$-genus? The bundles $V_{d,n}^\perp = \theta^*\gamma_{d,n} \to \mathrm{Gr}_{d,n}^{\mathrm{Spin}}$ have KO-Thom classes $\lambda_{V_{d,n}^\perp} \in KO(\mathrm{Th}(V_{d,n}^\perp))$, which assemble to a spectrum map

$$\mathsf{MT}\,\mathrm{Spin}(d) \xrightarrow{\lambda_{-d}} \Sigma^{-d}\mathsf{KO}$$

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[1] or rather its corollary about the surgery equivalence from the previous talk, see also [BER17, Thm. 2.3.4]

[2] two maps are weakly homotopic, if all precompositions with maps from a finite CW complex are homotopic, cf. [BER17, Def. 1.1.2]

(see [BER17, Par. 3.8.3]). The map in Theorem B is defined as the associated infinite loop map $\Omega^\infty \lambda_{-d} = \hat{A}_d$. It can be related to the classical $\hat{A}$-genus by Atiyah, Bott, and Shapiro [ABS64].

We will now prove Theorem B in the case $d = 2n = 6$ as in [BER17, Par. 4.3.1]. Consider the simply connected manifold $W = D^6$ (viewed as a spin cobordism $\emptyset \rightsquigarrow S^{2n-1}$) and set $W_k = W \# W_{k,1}$. We now apply the following results of Galatius and Randal-Williams

1. ψ is a homology equivalence if $g \gg 0$ (first block of this workshop, [GR14])
2. $\Omega^\infty \mathsf{MT}\,\mathrm{Spin}(6)$ is simply-connected, which can be extracted from [GR16].

Together they imply, that the map ψ in our diagram

$$\begin{tikzcd}
B\mathrm{Diff}_\partial(W_k) \arrow[r] \arrow[d, "\psi"] & B\,\mathrm{Aut}(\mathcal{R}^+(W_k)_h) \\
\Omega^\infty \mathsf{MT}\,\mathrm{Spin}(2n) \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

is acyclic, which finally gives us the dashed map (as discussed above), when passing to the homotopy colimit as $k \to \infty$.

Now there is the fibre bundle

$$\mathcal{R}^+(W)_h \longrightarrow E\,\mathrm{Aut}(\mathcal{R}^+(W)_h) \longrightarrow B\,\mathrm{Aut}(\mathcal{R}^+(W)_h)$$

which we can pull back using the newly obtained g. The fibre transport associated to this gives us the map $\rho\colon \Omega\Omega^\infty \mathsf{MT}\,\mathrm{Spin}(6) \to \mathcal{R}^+(W)_h$ from Theorem B.

1.3 Index Theory (again)—Further Ingredients for the Proof of Theorem B

Suppose a we have bundle $W^d \to E \to X$ of Riemannian spin manifolds. If we have fibrewise metrics g with some boundary conditions, we have a family of Dirac operators $\not{\mathfrak{D}}_g$ parametrized by X. This family defines[3] an index class

$$\mathrm{ind}(E, g) \in KO^{-d}(X)$$

It should be noted that X only needs to be paracompact in the model for KO-theory used there; also there are some technical assumptions needed for $W^d \to E \to X$.

In particular we get a homotopy class of a map $X \to \Omega^{\infty+d}\mathsf{KO}$. The following diagram compares the index with the $\hat{A}$-genus, that appeared in Theorem B:

$$\begin{tikzcd}
X \arrow[r, "\alpha_E"] \arrow[dr, "\mathrm{ind}(E,g)"'] & \Omega^\infty \mathsf{MT}\,\mathrm{Spin}(d) \arrow[d, "\hat{A}"] \\
& \Omega^{\infty+d}\mathsf{KO}
\end{tikzcd}$$

Here α_E is the Pontrjagin–Thom map of E. Indeed we have:

Theorem (Index-Theorem, [BER17, Thm. 3.8.4])**.** *$\mathrm{ind}(E, g)$ and $\hat{A}_d \circ \alpha_E$ are weakly homotopic.*

Remark. This generalises the classical Atiyah–Singer Index Theorem.

In the proof of Theorem B one now uses this index theorem and the so called relative index construction ([BER17, Cor. 3.5.2]), to get the statement about factorising $\Omega\hat{A}_6$.

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[3] see [BER17, Def. 3.2.4]

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